High Field Nonlinear Current

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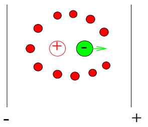
Debye-Onsager-Relaxation-Effect beyond Linear Response



Screening leads to retarding dipole force

Debye '23

$$E\,(1-\frac{\kappa e^2}{6\epsilon_0 T})$$

Onsager '27

$$E\,(1-\frac{\kappa e^2}{3(2+\sqrt{2})\epsilon_0 T})$$

Treatment within kinetic theory

$$\frac{\partial}{\partial t}f - eZ\,\mathbf{E}\frac{\partial}{\partial \mathbf{k}}f = I[f,\mathbf{E}]$$

Balance of current $\mathbf{J}(E,t) = nZ\frac{e}{m}\mathbf{p}(E,t)$
from $f(\mathbf{k},t) = f_0(\mathbf{k}-\mathbf{p}(\mathbf{E},t))$

$$\frac{\partial}{\partial t}\mathbf{p} - eZn(1+\frac{\delta E(E,t)}{E})\,\mathbf{E} = R(E,t)eZn\,\mathbf{J}(E,t)$$

Conductivity $$\sigma(E) = \frac{1+\frac{\delta E(E)}{E}}{R(E)}$$

Assumptions homogeneous (?) stationary (?) Maxwell (?) distribution $\quad E < \frac{\kappa T}{e}$

Classical BBGKY Hierarchy

One F_a and two - particle distribution function F_{ab}

$$\frac{\partial F_a}{\partial t} + \mathbf{v}\frac{\partial F_a}{\partial \mathbf{r}} + \frac{e_a}{m_a}\bar{\mathbf{E}}\frac{\partial F_a}{\partial \mathbf{v}} - S_a F_a = \sum_b \frac{n_b e_a e_b}{m_a}\frac{\partial}{\partial \mathbf{v}}\int d\mathbf{r}'\,d\mathbf{v}'\,F_{ab}(\mathbf{r},\mathbf{r}',\mathbf{v},\mathbf{v}')\frac{\partial}{\partial \mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

$$\frac{\partial F_{ab}}{\partial t} + \mathbf{v}\frac{\partial F_{ab}}{\partial \mathbf{r}} + \mathbf{v}'\frac{\partial F_{ab}}{\partial \mathbf{r}'} + \frac{e_a}{m_a}\bar{\mathbf{E}}\frac{\partial F_{ab}}{\partial \mathbf{v}} + \frac{e_b}{m_b}\bar{\mathbf{E}}\frac{\partial F_{ab}}{\partial \mathbf{v}'} - S_a F_{ab} - S_b F_{ab} = e_a e_b \frac{\partial}{\partial \mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\left(\frac{1}{m_a}\frac{\partial F_{ab}}{\partial \mathbf{v}} - \frac{1}{m_b}\frac{\partial F_{ab}}{\partial \mathbf{v}'}\right)$$
$$+\sum_c n_c e_c \int d\mathbf{r}''\,d\mathbf{v}''\left(\frac{e_a}{m_a}\frac{\partial}{\partial \mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{r}''|}\cdot\frac{\partial F_{abc}}{\partial \mathbf{v}} + \frac{e_b}{m_b}\frac{\partial}{\partial \mathbf{r}'}\frac{1}{|\mathbf{r}'-\mathbf{r}''|}\cdot\frac{\partial F_{abc}}{\partial \mathbf{v}'}\right)$$

For weakly coupled plasma we decouple

$$F_{ab} = F_a F_b + g_{ab} \quad, F_{abc} = F_a F_b F_c + F_a g_{bc} + F_b g_{ac} + F_c g_{ab}$$

Solve with a local Maxwellian distribution

$$F_a(\mathbf{r},\mathbf{v},t) = n_a(\mathbf{r},t)\left(\frac{m_a}{2\pi T}\right)^{3/2} \mathrm{e}^{\left[-\frac{m_a(\mathbf{v}-\mathbf{u_a})^2}{2T}\right]}$$
$$g_{ab}(\mathbf{r},\mathbf{r}',\mathbf{v},\mathbf{v}',t) = h_{ab}(\mathbf{r},\mathbf{r}',t)\left(\frac{m_a m_b}{4\pi^2 T^2}\right)^{3/2} \mathrm{e}^{\left[-\frac{m_a(\mathbf{v}-\mathbf{w_{ab}})^2}{2T}-\frac{m_b(\mathbf{v}'-\mathbf{w_{ba}})^2}{2T}\right]}$$

Collision integrals S_a with background assume

$$\int S_a f_a d\mathbf{v_a} = 0 \qquad \int \mathbf{v} S_a f_a d\mathbf{v} = \frac{1}{b_a m_a} n_a u_a$$
$$\int \mathbf{v} S_a g_{ab}(\mathbf{r},\mathbf{r}',\mathbf{v},\mathbf{v}',t)d\mathbf{v} = \frac{1}{b_a m_a} h_{ab} w_{ab}$$

With background

derive Onsager equations for potential ϕ

$$\Delta_a \phi_b(\mathbf{r_a},\mathbf{r_b}) = -4\pi\left[\sum_c e_c N_c h_{cb}(\mathbf{r_a},\mathbf{r_b}) + e_b\delta(\mathbf{r_a}-\mathbf{r_b})\right]$$

$$T(b_a+b_b)\Delta h_{ab}(\mathbf{r_a},\mathbf{r_b}) + e_a b_a \Delta_a\phi_b(\mathbf{r_a},\mathbf{r_b}) + e_b b_b \Delta_a \phi_a(\mathbf{r_b},\mathbf{r_a}) = (e_a b_a - e_b b_b)\mathbf{E}\cdot\nabla h_{ab}(\mathbf{r_a},\mathbf{r_b})$$

Solution with $\alpha = eE/T\kappa_e$ and $Z=1$:

$$\delta\mathbf{E} = -\frac{e^2\kappa_e}{3(1+\sqrt{2})\,T}\mathbf{E}\,F_H(\alpha)$$
$$F_H(\alpha) = \frac{3(1+\sqrt{2})}{\alpha^2}\left[\frac{1}{2}\sqrt{\alpha^2+2} - 1 + \frac{1}{\alpha}\arctan(\alpha) - \frac{1}{\alpha}\arctan(\frac{\alpha}{\sqrt{\alpha^2+2}})\right]$$

For $Z \neq 0$ linear response $$\delta\mathbf{E} = -\frac{e^2\kappa}{3T}\frac{Zq}{\sqrt{q}+1}\mathbf{E} \qquad q = \frac{b_e + Zb_i}{(1+Z)(b_e+b_i)}$$

Without background

With $S_a \equiv 0$ we get a complete different equation system

$$(T\Delta - e_a\mathbf{E}\cdot\nabla)h_{ab}(\mathbf{r_a},\mathbf{r_b}) + e_a\Delta_a\phi_b(\mathbf{r_b},\mathbf{r_a}) = 0$$
$$(T\Delta + e_b\mathbf{E}\cdot\nabla)h_{ab}(\mathbf{r_a},\mathbf{r_b}) + e_b\Delta_a\phi_a(\mathbf{r_a},\mathbf{r_b}) = 0$$

Solution $(Z=1)$

$$\delta\mathbf{E} = -\frac{e^2\kappa_e\sqrt{1+Z}}{6T}(Z+1)\mathbf{E}F_N(\alpha) \qquad \frac{\delta E}{E} = -\frac{e^2\kappa_e}{6\,T}\begin{cases} 2+o(E) \\ \frac{3\kappa T}{eE} + o(1/E)^2\end{cases}$$

Real-time Greenfunctions

Two independent correlation functions $1 = r,t,s,\ldots$
$G^>(1,2) = < a(1)a^+(2) >$ and $G^<(1,2) = < a^+(2)a(1) >$

Martin Schwinger Hierarchy for causal function $iG = \Theta_{12}G^> \pm \Theta_{21}G^<$

$$G_0^{-1}G(1,1') = \delta(1,1') + \int d3V(1,3)G_2(1,3,1',3^+) \equiv \int_c d3\Sigma(1,3)G(3,1')$$

with $G_0^{-1}(11') = \left(i\frac{\partial}{\partial t_1} - \frac{(\frac{\hbar}{i}\nabla_{x_1})^2}{2m} - \Sigma^{HF}(11')\right)\delta(1-1')$, using weakening of initial correlations $\lim_{t\to t_0}\Sigma G = 0$, one identifies the Keldysh contour

$$\int_c d\bar{1}\Sigma(1,\bar{1})G(\bar{1},1') = \int_{-\infty}^{+\infty} d\bar{1}\,\{\Sigma(1,\bar{1})G(\bar{1},1') - \Sigma^<(1,\bar{1})G^>(\bar{1},1')\}$$

Consequence is the Kadanoff and Baym equation

$$-i\left(G_0^{-1}G^< - G^<G_0^{-1}\right)(t_1,t_2) = \int_{t_0}^{t_1}\Sigma^>G^< + \int_{t_0}^{t_2}G^<\Sigma^> - \int_{t_0}^{t_1}G^>\Sigma^< - \int_{t_0}^{t_2}\Sigma^<G^> + \int_{t_1}^{t_2}[\Sigma^<,G^<]\,.$$

Time diagonal equation reads (Levinson equation) $G^<(t,t) = f(t)$

$$i[G_0^{-1},f(t)] = \int_0^{t-t_0} d\tau\left(\{G^<(t-\frac{\tau}{2},\tau),\Sigma^>(t-\frac{\tau}{2},-\tau)\} - \{G^>,\Sigma^<\}\right)$$

Gauge invariance - The problem of ansatz

Fourier-transform of the coordinates $x = r_1 - r_2$

$$\bar{G}(k,X) = \int dx\;\mathrm{e}^{\left\{\frac{i}{\hbar}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda\, x_\mu[k^\mu + \frac{e}{c}A^\mu(X+\lambda x)]\right\}} < \Psi(R+\frac{r}{2})\Psi^+(R-\frac{r}{2}) >$$

such that Gauge transform becomes invariant

Rule: $\quad G(p,T) = G(k-eET,T) = \bar{G}(k,T) = \bar{G}(p+eET,T)$

Express Green's function into distribution function and spectral one, GKB ansatz [P. Lipavsky, V. Spicka, B. Velicky, Phys. Rev B43 (86) 6933]

$$G^<(\mathbf{k},\tau,\mathbf{r},t) = \exp\left\{-\frac{i}{\hbar}\left(\epsilon_k\tau + \frac{e^2E^2}{24m}\tau^3\right)\right\}f\left(\mathbf{k}-\frac{e\mathbf{E}|\tau|}{2},\mathbf{r},t-\frac{|\tau|}{2}\right)$$

- it has the correct spectral properties
- for high field transport it is gauge invariant
- it preserves causality
- the quantum kinetic equations derived with GKB coincide with those obtained with the density matrix technique
- it reproduces the Debye-Onsager relaxation effect

Dynamically screened approximation

Self energy by a sum of all ring diagrams in exact time convolutions

$$\Sigma_a^<(k,t,t') = \int\frac{dq}{(2\pi\hbar)^3}\mathcal{V}_{aa}^<(q,t,t')G_a^<(k-q,t,t')$$

where the dynamical potential is $$\mathcal{V}_{aa}^<(q,t,t') = \sum_{dc}V_{ad}(q)\mathcal{L}_{dc}^<(q,t,t')V_{ca}(q)$$

$$\mathcal{L}_{ab}^<(q,t,t') = \delta_{ab}\int d\bar{t}d\bar{\bar{t}}\,(\mathcal{E}^r)^{-1}(q,t,\bar{t})L_{aa}{}^<(q,\bar{t},\bar{\bar{t}})\,(\mathcal{E}^a)^{-1}(q,\bar{\bar{t}},t')$$

Here L is the free density fluctuation $$L_{aa}^<(q,t,t') = \int\frac{dp}{(2\pi\hbar)^3}G_a^<(p,t,t')G_a^>(p-q,t',t)$$

and $\mathcal{E}^{r/a}$ the retarded/ advanced dielectric function

$$\mathcal{E}^{r/a}(q,t,t') = \delta(t-t') \pm i\Theta[\pm(t-t')]\sum_b V_{bb}(q)(L^>(q,t,t') - L^<(q,t,t'))$$

Field dependent Lenard-Balescu equation

$$\frac{\partial}{\partial T}f_a + e\mathbf{E}\frac{\partial}{\partial\,\mathbf{k_a}}f_a = 2\sum_b\left(I_a^{\rm in}(\mathbf{k},t) - I_a^{\rm out}(\mathbf{k},t)\right)$$

$$I_a^{in}(k,t) = \int\frac{dq}{(2\pi\hbar)^3}V_{ab}^2(q)\int_0^\infty d\tau\int\frac{d\omega}{2\pi}\cos\left[(\epsilon_{k-q}^a - \epsilon_k^a - \omega)\tau + \frac{e_aEq\tau^2}{2m_a}\right]$$
$$\times\frac{L_{bb}^<(q,\omega,t-\frac{1}{2}\tau)}{\left|\mathcal{E}(q,\omega,t-\frac{1}{2}\tau)\right|^2}f_a(k-q-e_aE\tau,t-\tau)(1-f_a(k-e_aE\tau,t-\tau))$$

$$L_{bb}{}^<(q,\omega,t) = -2\int\frac{dp}{(2\pi\hbar)^3}\int_0^\infty d\tau\cos\left[(\omega-\epsilon_p^b+\epsilon_{p+q}^b)\tau + \frac{e_bEq\tau^2}{2m_b}\right]$$
$$\times\; f_b(p+q,t-\frac{1}{2}\tau)(1-f_b(p,t-\frac{1}{2}\tau))$$

Static approximation

Beyond Boltzmann finite duration of collisions and field effects on scattering

$$\frac{\partial}{\partial T}f_a + e\mathbf{E}\frac{\partial}{\partial\,\mathbf{k_a}}f_a = \frac{2s_b}{\hbar^2}\sum_b\int\frac{d\mathbf{k}_a' d\mathbf{k_b} d\mathbf{k}_b'}{(2\pi\hbar)^6}\delta(\mathbf{k_a}+\mathbf{k_b}-\mathbf{k}_a'-\mathbf{k}_b')$$
$$\times\; V_{ab}^2(\mathbf{k_a}-\mathbf{k}_a')\int_0^\infty d\tau\cos\left\{\frac{(\epsilon_a+\epsilon_b-\epsilon_a'-\epsilon_b')\tau}{\hbar} - \frac{\mathbf{E}\tau^2}{2\hbar}\left(\frac{e_a\mathbf{k_a}}{m_a}+\frac{e_b\mathbf{k_b}}{m_b}-\frac{e_a\mathbf{k}_a'}{m_a}-\frac{e_b\mathbf{k}_b'}{m_b}\right)\right\}$$
$$\times\;\{f_a'(k_a'-e_a'E\tau,T-\tau)f_b'(1-f_a)(1-f_b) - f_af_b(1-f_a')(1-f_b')\}$$

analytical result (7-dim. integral) $$\frac{\partial}{\partial t}< p > - neE\left(1-\frac{\delta E}{E}\right) = 0$$

classical limit $(\hbar\to 0)$

$$\frac{\delta E_a}{E} = -\frac{\kappa e_a^2}{6\epsilon_0 T}\frac{Z(1+\frac{m_e}{m_i}Z)}{(1+Z)(1+\frac{m_e}{m_i})}\mathcal{F}\left(\frac{eE}{T\kappa}\frac{Z(1+\frac{m_e}{m_i}Z)}{1+\frac{m_e}{m_i}}\right)$$
$$= -\frac{e^2\kappa_e}{6\,T}\begin{cases}\frac{1}{2}+o(E) \\ \frac{3\kappa T}{2eE}+o(1/E)^2\end{cases}\qquad \text{for } Z=1$$

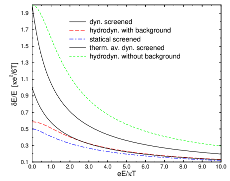
Relaxation effect for dynamical screening

For single charged ions and big mass differences, twice the Debye result

$$\frac{\delta E_a}{E} = -\frac{\kappa e_a^2}{6T}\mathcal{F}[\frac{eE}{\kappa T}] = -\frac{e^2\kappa_e}{6\,T}\begin{cases}2+o(E) \\ \frac{3\kappa T}{\sqrt{2}eE}+o(1/E)^2\end{cases}$$

Approximative treatment: Replace dynamical screening $\overline{\epsilon(\omega,q)^{-2}}$ by thermal averaging $(1+\kappa^2V_{aa}(q)/4\pi)^{-1}$, replace V_sV_s by V_sV, Debye result:

$$\frac{\delta E_a}{E} = -\frac{\kappa e_a^2}{6T}\mathcal{F}_{\rm dyn}[\frac{eE}{\kappa T}] = -\frac{e^2\kappa_e}{6\,T}\begin{cases}1+o(E) \\ \frac{3\kappa T}{eE}+o(1/E)^2\end{cases}$$



Asymmetric dynamical screening

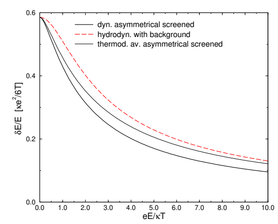
Particles b (ions) dynamically screened, a statically screened (Onsager treatment-background), for electron-ion plasma $q = 1/(Z+1)$:

$$\frac{\delta E_a}{E} = \frac{\delta E^{\rm dyn}}{E}\frac{Zq}{\sqrt{q}+1} = -\frac{\kappa e^2}{3T}\frac{Zq}{\sqrt{q}+1} + o(E)$$

The low field limit agrees with the hydrodynamical result exactly.
Approximative treatment

$$\frac{\delta E_a}{E} = -\frac{\kappa e_a^2}{6T}\mathcal{F}_{\rm asy}^{\rm dyn}[\frac{eE}{\kappa T}] = -\frac{e^2\kappa_e}{6\,T}\begin{cases}2-\sqrt{2}+o(E) \\ \frac{3\kappa T}{2eE}+o(1/E)^2\end{cases}$$

The thermally averaged approximation agrees with the hydrodynamical approximation in the low and high field limit.



Summary: Phys. Rev.**E 62** (2000) 6135

- Nonlinear current lowered by relaxation effect describing the influence of screening on the motion of particles
- Classical hydrodynamical approach (beyond linear response)
  - Debye result of symmetric system without infinitesimal background
  - Onsager result for background or asymmetrically treated systems
- Kinetic theory:
  - Statical screening: Half Debye result
  - Dynamical screening: Twice Debye result
  - Thermally averaged dynamical screening: Debye result
  - Asymmetrically dynamical screening: Onsager result
  - Asymmetrically dynamical screening: Onsager result for low fields and large field behaviour as classical hydrodynamical approach