Three Particles in Magnetic Fields

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The model

Hamiltonian of three interacting particles in magnetic field
$\mathbf{A}_\alpha = \frac{B}{2}\{-y, x, 0\}$

$$\hat{H} = -\frac{\hbar^2}{2m^*}\sum_{\alpha=1}^{3}\left(\frac{\partial}{\partial \mathbf{r}_\alpha} - i\frac{e}{\hbar c}\mathbf{A}_\alpha\right)^2 + \frac{\kappa}{2}[(\mathbf{r}_1-\mathbf{r}_2)^2 + (\mathbf{r}_1-\mathbf{r}_3)^2 + (\mathbf{r}_2-\mathbf{r}_3)^2]$$

$\kappa = m^*\omega_0^2/3$ with ω_0 specific frequency of electron's relative vibration
In 2D it can be diagonalized $\hat{H} = \hat{H}_\mathbf{r} + \hat{H}_\xi + \hat{H}_\eta$ with coordinates

$$\mathbf{r} = \frac{1}{\sqrt{3}}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3), \quad \xi = \frac{1}{\sqrt{2}}(\mathbf{r}_1-\mathbf{r}_2), \quad \eta = \sqrt{\frac{2}{3}}\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2} - \mathbf{r}_3\right)$$

with three separate Schrödinger equations

$$\hat{H}_\rho \Psi_{n_i,m_i} = E_{n_i,m_i}\Psi_{n_i,m_i}$$

solution $[\Gamma = \sqrt{1+\frac{4\omega_0^2}{\omega_B^2}}/2l_B^2]$:

$$\psi_{n_i,m_i}(\rho,\varphi) = \left[\frac{n!}{\pi(n_i+|m_i|)!}\right]^{1/2}\Gamma^{\frac{|m|+1}{2}}e^{im_i\varphi}\rho^{|m_i|}\mathrm{e}^{-\Gamma\rho^2}L_{n_i}^{|m_i|}\left(\Gamma\rho^2\right)$$

with cyclotron frequency: $\omega_B = \frac{eB}{m^*c}$ and magnetic length: $l_B^2 = \frac{\hbar c}{eB}$
Total Energies
$n_0 = n_1 + \frac{1}{2}(|m_1|+m_1)$, $n = 2(n_2+n_3)+|m_2|+|m_3|$ and $m = m_2+m_3$

$$E(n_0,n,m) = \hbar\omega_B(n_0+\frac{1}{2}) + \frac{\hbar\omega_B}{2}\sqrt{1+\frac{4\omega_0^2}{\omega_B^2}}(n+2) + \frac{\hbar\omega_B}{2}m.$$

Total wave function $\psi_{n_1,m_1}(r,\omega_0=0)\psi_{n_2,m_2}(\xi)\psi_{n_3,m_3}(\eta)$

Wave functions and correlators

In high magnetic fields spins align building quartet states $|S=\frac{3}{2},s_z\rangle$, spinors are symmetric

$$\begin{aligned}
|3/2,3/2\rangle &= \alpha_1\alpha_2\alpha_3\\
|3/2,1/2\rangle &= \frac{1}{\sqrt{3}}\{\alpha_1\alpha_2\beta_3+\alpha_1\beta_2\alpha_3+\beta_1\alpha_2\alpha_3\}\\
|3/2,-1/2\rangle &= \frac{1}{\sqrt{3}}\{\alpha_1\beta_2\beta_3+\beta_1\beta_2\alpha_3+\beta_1\alpha_2\beta_3\}\\
|3/2,-3/2\rangle &= \beta_1\beta_2\beta_3
\end{aligned}$$

antisymmetric orbital part of the wave function ($\Psi=0$ for m_2 =even)

$$\begin{aligned}
\Psi^Q &= C\Psi_{n_1,m_1}(\mathbf{r};\omega_o=0)\\
&\times\left\{\Psi_{n_2,m_2}(\xi)\Psi_{n_3,m_3}(\eta)+\Psi_{n_2,m_2}(\tilde{\xi})\Psi_{n_3,m_3}(\tilde{\eta})+\Psi_{n_2,m_2}(\tilde{\tilde{\xi}})\Psi_{n_3,m_3}(\tilde{\tilde{\eta}})\right\}
\end{aligned}$$

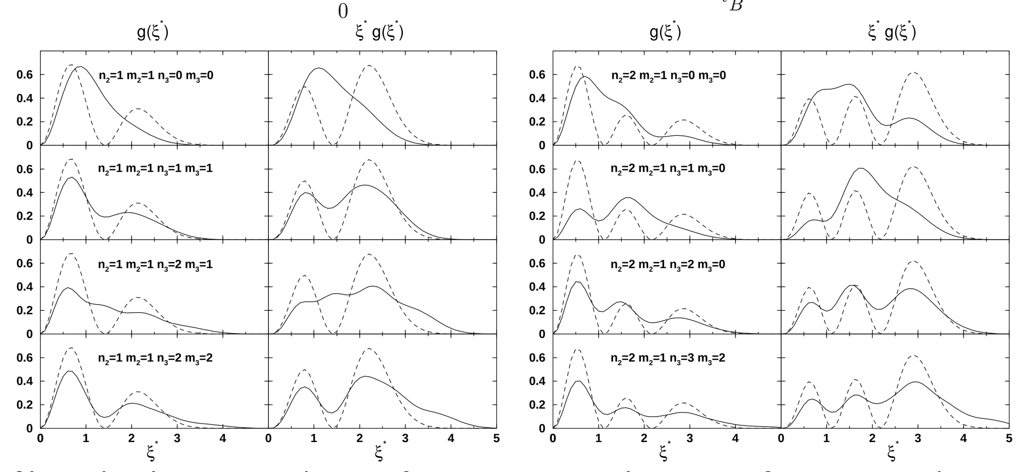
with notation

$$\begin{aligned}
\tilde{\xi} &= \frac{\mathbf{r_2}-\mathbf{r_3}}{\sqrt{2}} = \frac{\sqrt{3}}{2}\eta - \frac{\xi}{2}, \quad \tilde{\eta} = \sqrt{\frac{2}{3}}\left(\frac{\mathbf{r_2}+\mathbf{r_3}}{2}-\mathbf{r_1}\right) = -\frac{\eta}{2}-\frac{\sqrt{3}}{2}\xi,\\
\tilde{\tilde{\xi}} &= \frac{\mathbf{r_3}-\mathbf{r_1}}{\sqrt{2}} = -\frac{\sqrt{3}}{2}\eta - \frac{\xi}{2}, \quad \tilde{\tilde{\eta}} = \sqrt{\frac{\mathbf{2}}{\mathbf{3}}}\left(\frac{\mathbf{r_1}+\mathbf{r_3}}{\mathbf{2}}-\mathbf{r_2}\right) = -\frac{\eta}{\mathbf{2}}+\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\xi
\end{aligned}$$

- permit only odd values of m_2 as in fractional quantum Hall experiments
- wave function Ψ^Q has property that $\Psi^Q(\mathbf{r_1},\mathbf{r_2},\mathbf{r_3})|_{\mathbf{r_i}\to\mathbf{r_j}} = 0$ for arbitrary $i\neq j = 1,2,3$ corresponding to Jastrow prefactor $f(\mathbf{r_i}-\mathbf{r_j})$ in Laughlin's ground state wave function [3,4]

Laughlin's wave function ?

Pair-correlation function $\int_0^{2\pi} d\phi_\xi \int d^2r \int d^2\eta\,\left|\Psi^Q\right|^2 = \frac{\sqrt{1+\frac{4\omega_0^2}{\omega_B^2}}}{l_B^2}g(|\xi^*|)$



Normalized pair-correlation function vs. distance of two particles $\xi^* = \xi^* = \left(1+\frac{4\omega_o^2}{\omega_B^2}\right)^{1/4}\xi/\sqrt{2}l_B$
dotted lines: pair correlation function only from the diagonal elements

- shell structure appears due to presence of third particle
- number of maxima (shells) equals to n_2+1, onset of liquid like behavior

Small ξ expansion

$$\begin{aligned}
g(|\xi^*|) &= a|\xi^*|^{2m_2} + (b+c|\xi^*|^{2m_2})|\xi^*|^2 + o(|\xi^*|^3)\\
&= \begin{cases}(a+b)|\xi^*|^2, & m_2=1\\ b|\xi^*|^2+a|\xi^*|^{2m_2}, & m_2>1\end{cases} + o(|\xi^*|^3)
\end{aligned}$$

- characteristic property of Laughlin wave function: $g(z)\sim z^{2m}$
- characteristic property of Wigner crystal $\sim 1-\exp(-az^2)\propto z^2$
- we obtain exactly Wigner crystal behavior and not Laughlin !

However, Laughlin exchange correlation hole becomes transparent if we use only the first diagonal term in
$|\Psi^Q|^2 \propto C^2|\Psi_{n_1,m_1}(r;\omega_0=0)|^2|\Psi_{n_2,m_2}(\xi)|^2|\Psi_{n_3,m_3}(\eta)|^2$, analytical result

$$\begin{aligned}
g^{\mathrm{diag1}}(|\xi^*|) &= \frac{n_2!}{(n_2+|m_2|)!}\xi^{*2|m_2|}\left(L_{n_2}^{|m_2|}(\xi^{*2})\right)^2\mathrm{e}^{-\xi^{*2}}\\
&= \frac{(n_2+|m_2|)!}{n_2!m_2!^2}\xi^{*2|m_2|}\left(1+o(\xi^2)\right)
\end{aligned}$$

- typical $z^{2|m_2|}$ behavior of Laughlin states appears
- we consider quantum number $|m_2|$ as the filling factor
- approximation leading to this behavior consists in neglecting certain crossed terms which might become exact in many-particle limit

Analytical results for observables

use all three diagonal elements
$|\Psi^{Q\mathrm{diag}}|^2 = C^2|\Psi_1|^2\left(|\Psi_2|^2|\Psi_3|^2+|\tilde{\Psi}_2|^2|\tilde{\Psi}_3|^2+|\tilde{\tilde{\Psi}}_2|^2|\tilde{\tilde{\Psi}}_3|^2\right)$

$n_2m_2n_3m_3$	$1/C$	dist.	area
1100	2.24	2.49	1.13
	3	2.5	1.5
1111	3.37	4.00	2.25
	3	4	6
1121	1.83	5.00	7.35
	3	5	9
1122	4.81	5.49	11.03
	3	5.5	10.5
2100	2.79	3.50	2.11
	3	3.5	2.25
2110	1.12	4.49	8.07
	3	4.5	6.75
2111	1.83	5.00	7.35
	3	5	9
2120	3.22	5.48	5.07
	3	5.5	11.25
2121	2.67	5.99	9.94
	3	6	13.5
2122	3.10	6.46	25.10
	3	6.5	15.75
2130	2.17	6.44	16.12
	3	6.5	15.75
2131	3.26	6.92	10.88
	3	7	18
2132	3.28	7.26	28.59
	3	7.5	20.25
2133	2.27	7.68	14.47
	3	8	22.5

Mean distance of particles

$$\begin{aligned}
&\langle\frac{(\mathbf{r}_1-\mathbf{r}_2)^2}{2}\rangle\\
&= \int d^2\xi d^2r d^2\eta\,\xi^2\left|\Psi^{Q\mathrm{diag}}\right|^2\\
&= \frac{3}{2}\frac{C^2}{\Gamma}[(1+2n_2+|m_2|)+(1+2n_3+|m_3|)]
\end{aligned}$$

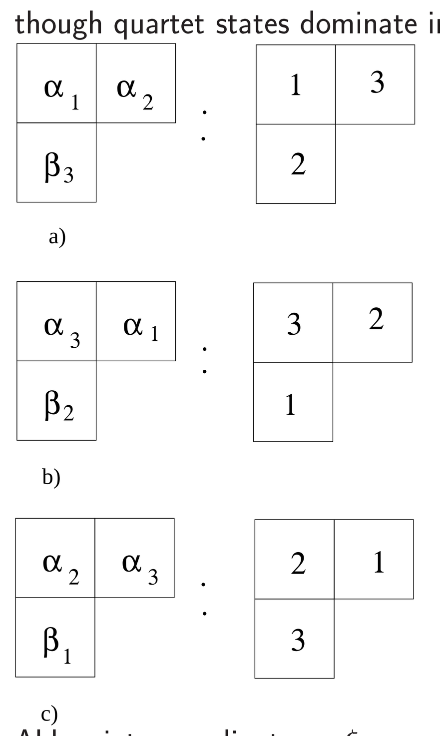
Area spanned by 3 particles

$$\begin{aligned}
&\langle\frac{1}{4}|(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_3-\mathbf{r}_1)|^2\rangle\\
&= \int d^2\xi d^2r d^2\eta\frac{3}{4}|\xi\times\eta|^2\left|\Psi^{Q\mathrm{diag}}\right|^2\\
&= \frac{9}{16}\frac{C^2}{\Gamma^2}[(1+2n_2+|m_2|)(1+2n_3+|m_3|)]
\end{aligned}$$

Normalization constant C, dimensionless mean distance and area for different quantum numbers. The corresponding second lines give the analytical values of diagonal terms respectively. This becomes better for larger quantum numbers.

Doublet states

though quartet states dominate in high fields, also doublet states possible



Young diagrams contributing to spin state $|S=1/2,s_z=1/2\rangle$ (left) and contributions to spatial wave function (right). Indices in one row are symmetrized and indices in one column are antisymmetrized.. Diagrams for $|S=1/2,s_z=-1/2\rangle$ are obtained by interchanging $\alpha\leftrightarrow\beta$.

Abbreviate coordinates r,ξ,η as (1,23), three orbital wave function read

$$\begin{aligned}
\Psi_a^D &= (2,13)+(2,31)-(1,23)-(1,32)\\
\Psi_b^D &= (1,32)+(1,23)-(3,21)-(3,12)\\
\Psi_c^D &= (3,21)+(3,12)-(2,31)-(2,13).
\end{aligned}$$

final doublet wave function for m_2 odd

$$\begin{aligned}
\Psi^D(\mathbf{r_1},\mathbf{r_2},\mathbf{r_3}) &= C_1\Psi_{n_1,m_1}(r;\omega_0=0)\\
&\times\big\{(\beta_1\alpha_2\alpha_3+\alpha_1\beta_2\alpha_3-2\alpha_1\alpha_2\beta_3)\Psi_{n_2,m_2}(\xi)\Psi_{n_3,m_3}(\eta)\\
&+(\alpha_1\beta_2\alpha_3+\alpha_1\alpha_2\beta_3-2\beta_1\alpha_2\alpha_3)\Psi_{n_2,m_2}(\tilde{\xi})\Psi_{n_3,m_3}(\tilde{\eta})\\
&+(\beta_1\alpha_2\alpha_3+\alpha_1\alpha_2\beta_3-2\alpha_1\beta_2\alpha_3)\Psi_{n_2,m_2}(\tilde{\tilde{\xi}})\Psi_{n_3,m_3}(\tilde{\tilde{\eta}})\big\}
\end{aligned}$$

while for m_2 even

$$\begin{aligned}
\Psi^D(\mathbf{r_1},\mathbf{r_2},\mathbf{r_3}) &= C_2\Psi_{n_1,m_1}(r;\omega_0=0)\\
&\times\big\{(\alpha_1\beta_2\alpha_3-\beta_1\alpha_2\alpha_3)\Psi_{n_2,m_2}(\xi))\Psi_{n_3,m_3}(\eta)\\
&+(\alpha_1\alpha_2\beta_3-\alpha_1\beta_2\alpha_3)\Psi_{n_2,m_2}(\tilde{\xi})\Psi_{n_3,m_3}(\tilde{\eta})\\
&+(\beta_1\alpha_2\alpha_3-\alpha_1\alpha_2\beta_3)\Psi_{n_2,m_2}(\tilde{\tilde{\xi}})\Psi_{n_3,m_3}(\tilde{\tilde{\eta}})\big\}.
\end{aligned}$$

- Doublet $|\Psi^D|^2$ differs from quartet state by the coefficients in front of the crossed terms
- Contributions of crossed terms in quartet states smaller than those in doublet states, maxima in pair correlation function are more pronounced in former case

Summary [8]

1. Three-particle problem in magnetic fields is essential to understand phenomenon of the fractional quantum Hall effect [4]
2. We have solved exactly problem of three identical particles interacting via spring potential in 2D space in the presence of external magnetic field
3. The wave function acquires the Jastrow prefactor even in the case of attractive potential as consequence of anti-symmetric character of total wave function
4. Pair correlation function, area of three-particle cluster and mean distance show quantization
5. Pair correlation function exhibits shell structure, presence of third particle induces correlations similar to liquid correlations
6. Analogous to the situation of oscillating density profiles around a quantum dot if two interacting electrons are placed in dot [1].
7. Exchange correlation hole at small distances shows behavior of Wigner lattices rather than Laughlin behavior
8. Neglecting certain cross terms, we obtain exactly the Laughlin exchange correlation hole, corresponding filling factor is identified with quantum number appearing in exact solution of the model
9. Three-particle clustering possible mechanism for observed metal-insulator transition [2], discussed in [5–7]

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