

Retarded versus Time-Nonlocal Quantum Kinetic Equations

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The finite duration of the collisions in Fermionic systems as expressed by the retardation time in non-Markovian Levinson-type kinetic equations is discussed in the quasiclassical limit. We separate individual contributions included in the memory effect resulting in (i) off-shell tails of the Wigner distribution, (ii) renormalization of scattering rates, (iii) renormalization of the single-particle energy, (iv) collision delay, and (v) related nonlocal corrections to the scattering integral. In this way we transform the Levinson equation into the Landau–Silin equation extended by the nonlocal corrections known from the theory of dense gases. The derived nonlocal kinetic equation unifies the Landau theory of quasiparticle transport with the classical kinetic theory of dense gases. The space-time symmetry is discussed versus particle-hole symmetry and a solution is proposed which transforms these two exclusive pictures into each other. © 2001 Elsevier Science

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1. INTRODUCTION

The generalization of the Boltzmann equation (BE) towards dense interacting quantum systems is a still demanding and unsolved task. In dense systems, the mean time between successive collisions becomes comparable to the collision duration. In this case all particles cannot be described as a system of weakly interacting quasiparticles, but a set of particles bound in two-particle correlations has to be accounted for in a distinct manner. In the spirit of chemical reactions, one can introduce the quasiparticle density, n_f (a density of single-atom molecules), and the density of particles in the correlated states, n_c (twice the density of bi-atomic molecules). The total density n is their sum

$$n = n_f + n_c. \quad (1)$$

In parallel with the partial pressure of molecules, $\mathcal{P} = k_B T(n_f + \frac{1}{2}n_c)$, and the Guldberg–Waage law of mass action law, $n_c = K n_f^2$, even a small fraction of the correlated density, $n_c \ll n_f$, contributes to the second virial coefficient, $\mathcal{P} \approx k_B T(n - \frac{1}{2}K n_f^2)$. The correlated density thus plays an important role in the thermodynamic behavior of the system.

While the equation of state of the ideal gas, $\mathcal{P} = k_B T n$, does not reflect any microscopic properties of particles in the system, the second virial coefficient directly follows from the particle–particle interaction. In the early time of statistical mechanics, the experimental second virial coefficient

has been used to deduce interaction potentials. Surprisingly, the firmly established concept of the equilibrium virial expansion has never found a corresponding position in the theory of nonequilibrium systems although a number of attempts have been made to modify the BE so that its equilibrium limit will cover at least the second virial coefficient [1–3]. The achieved corrections to the BE have a form of gradient or nonlocal contributions to the scattering integral. For a hard-sphere gas the nonlocal correction is trivial; therefore the main theoretical focus was on the statistical correlations [4–15]. It turned out that the treatment of higher order contributions is far from trivial as the dynamical statistical correlations result in divergences that are cured only after a re-summation of an infinite set of contributions. Naturally, this re-summation leads to nonanalytic density corrections to the scattering integral [11–15]. The moral of these hard-sphere studies is that beyond the nonlocal corrections one must sum up an infinite set of contributions; the plain expansion leads to incorrect results. From this point of view, the approximate statistical virial corrections implemented by [16], although possibly reasonable, are not sufficiently justified by the theory.

Real particles do not interact like hard spheres, in particular when their de Broglie wave lengths are comparable with the potential range. An effort to describe the virial corrections for more realistic systems resulted in various generalizations of Enskog's equation [1, 3, 17–40]. By closer inspection one finds that all tractable quantum theories deal exclusively with the nonlocal corrections. The statistical correlations in quantum systems would require an adequate solution of three-particle collisions (say from Fadeev equations) which are now intensively being studied [41]. A systematic incorporation of three-particle collisions into the kinetic equation, however, is not yet fully understood; therefore we discuss only binary processes.

An alternative way to describe the correlation has been developed for Fermi systems at very low temperatures. Since the Pauli principle excludes all but the zero-angle scattering channels, the correlation can be re-cast into a renormalization of the single-particle energies known as the quasiparticle energies. While quasiparticles behave in many aspects like free particles, the density dependence of the quasiparticle energies results in nontrivial thermodynamic properties. In accordance with the focus either on zero-angle of finite-angle scattering, in the literature one can find two distinct classes of quantum kinetic equations.

The zero-angle class consists of equations determining the time evolution of the momentum and space dependent quasiparticle distribution function $f(k, r, t)$. This leads to kinetic equations of the Landau–Silin type

$$\frac{\partial f}{\partial t} + \frac{\partial \varepsilon}{\partial k} \frac{\partial f}{\partial r} - \frac{\partial \varepsilon}{\partial r} \frac{\partial f}{\partial k} = z((1 - f)\sigma_{\varepsilon}^< - f\sigma_{\varepsilon}^>). \quad (2)$$

Here the drift term is characterized by the quasiparticle energies $\varepsilon(k, r, t)$ and the collision integral consists of scattering-in and -out terms which are summarized in the selfenergy as functionals of the quasiparticle distribution function, $\sigma^{\lessgtr}[f]$. The collision rates are reduced by the wave function renormalization, which is given by the frequency derivative of the real part of the selfenergy σ at the pole value $z^{-1} = 1 - \frac{\partial \sigma}{\partial \omega}|_{\omega=\varepsilon}$. This kinetic equation is local in time and space; i.e., it describes the collisions as instantaneous point-like events. Similarly to the BE, the local collisions yield no explicit virial corrections. The virial corrections are thus covered exclusively by the quasiparticle energy while no correlated density appears.

The finite-angle class are equations which include the finite duration of collisions; i.e., they have non-Markovian scattering integrals. These kinetic equations are usually developed for the reduced density matrix or the Wigner function $\rho(k, r, t)$. The Wigner distribution obeys a Levinson-type of kinetic equation [42, 43]

$$\frac{\partial \rho_t}{\partial t} + \frac{\partial \varepsilon^{\text{hf}}}{\partial k} \frac{\partial \rho_t}{\partial r} - \frac{\partial \varepsilon^{\text{hf}}}{\partial r} \frac{\partial \rho_t}{\partial k} = 2 \text{Im} \int_{-\infty}^t d\bar{t} G_{t,\bar{t}}^R [(1 - \rho_{\bar{t}})\Sigma_{\bar{t},t}^< - \rho_{\bar{t}}\Sigma_{\bar{t},t}^>]. \quad (3)$$

Here the drift term is essentially determined by the Hartree–Fock mean field, $\epsilon^{\text{hf}} = k^2/2m + \sigma_{\text{hf}}$, and the double-time selfenergies $\Sigma_{\bar{t},t}^{\approx}$ are given as functionals of the Wigner distribution. Explicit time arguments of the collision integral, $\bar{t} < t$, show the retardation which requires us to include the propagation, $G_{\bar{t},t}^R$, from the retarded time \bar{t} to the time t of balancing changes. Having the non-Markovian form, this kinetic equation is capable to describe finite duration effects and corresponding virial corrections.

The Landau–Silin and Levinson equations have common features with the original BE. In particular, they express the balance between the drift given by the gradient terms on the left hand side and the dissipation given by the scattering integral on the right hand side. On the other hand, there is a remarkable difference in the actual physical contents of these common ingredients, namely $\rho \neq f$, $\varepsilon \neq \epsilon^{\text{hf}}$, and the right hand sides of (2) and (3) are rather different.

In the present paper we argue that the differences between the phenomenological quasiparticle picture, Eq. (2), and the BBGKY-type¹ equation (3) can be characterized in terms of the retardation of the collision integral of (3). We will show that there are various contributions to the total retardation, each responsible for different features seen in the quasiparticle picture extended by nonlocal corrections.

If both approaches are equivalent, the Levinson equation (3) must contain terms on the collisional side which should be possible to rearrange into gradients such that the quasiparticle energies of the drift side in the Landau–Silin equation (2) are obtained. The mutual relation between the Wigner distribution ρ and the quasiparticle distribution f is of essential importance for such a rearrangement.

In Section II we show that a part of the retardation of the Levinson equation (3) describes the off-shell motion of particles. This off-shell motion can be eliminated from the kinetic equation which requires to introduce an effective distribution (the quasiparticle distribution f) from which the Wigner distribution ρ can be constructed (45)

$$\rho = f + \int \frac{d\omega}{2\pi} \frac{\wp}{\omega - \varepsilon} \frac{\partial}{\partial \omega} ((1 - f)\sigma_{\omega}^< - f\sigma_{\omega}^>). \quad (4)$$

This relation is the extended quasiparticle picture derived for small scattering rates. The limit of small scattering rates has been first introduced by Craig [44]. An inverse functional $f[\rho]$ has been constructed in [45]. For equilibrium nonideal plasmas this approximation has been employed by [46, 47] and under the name of the generalized Beth–Uhlenbeck approach has been used by [48] in nuclear matter for studies of the correlated density. The authors in [49] have used this approximation with the name extended quasiparticle approximation for the study of the mean removal energy and high-momenta tails of Wigner’s distribution. The nonequilibrium form has been derived finally as the modified Kadanoff and Baym ansatz [50]. We will call it extended quasiparticle approximation.

We show that the retardation is responsible for gradient terms by which the mean-field drift of the Levinson equation (3) differs from the quasiparticle drift of the Landau–Silin equation (2). In Section III we discuss the remaining parts of the retardation, a piece which is compensated by the decay of internal propagators, and a piece which describes the collision delay. Special emphasis is put on internal double counts of correlations in the Levinson equation. Related to this question we present in Appendix C a discussion of a common pitfall found in the literature. In Section IV we complete the kinetic equation with the finite collision delay and provide related nonlocal corrections in space. Our main result is the nonlocal kinetic equation derived here in an alternative way to [51, 52],

$$\frac{\partial f_a}{\partial t} + \frac{\partial \varepsilon_a}{\partial k} \frac{\partial f_a}{\partial r} - \frac{\partial \varepsilon_a}{\partial r} \frac{\partial f_a}{\partial k} = \sum_b \int \mathcal{P} \{ f'_a f'_b (1 - f_a)(1 - f_b) - f_a f_b (1 - f'_a)(1 - f'_b) \} \quad (5)$$

¹ Equations derived from the Born–Bogoliubov–Green–Kirkwood–Yvon (BBGKY) hierarchy of reduced densities.

with $f'_a = f_a(k - q - \Delta_K, r - \Delta_3, t - \Delta_t)$, $f'_b = f_b(p + q - \Delta_K, r - \Delta_4, t - \Delta_t)$, $f_a = f_a(k, r, t)$, and $f_b = f_b(p, r - \Delta_2, t)$. The differential cross section $\mathcal{P} \sim |t^R|^2$ is proportional to the square of the amplitude of the T-matrix. All nonlocal corrections are given by derivatives of the scattering phase shift $\phi = \text{Im} \ln t^R(\omega, k, p, q, t, r)$ [51],

$$\begin{aligned} \Delta_t &= \frac{\partial \phi}{\partial \omega}, & \Delta_E &= -\frac{1}{2} \frac{\partial \phi}{\partial t}, & \Delta_K &= \frac{1}{2} \frac{\partial \phi}{\partial r}, & \Delta_3 &= -\frac{\partial \phi}{\partial k}, \\ \Delta_2 &= \frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial k}, & \Delta_4 &= -\frac{\partial \phi}{\partial k} - \frac{\partial \phi}{\partial q}. \end{aligned} \quad (6)$$

We discuss in Section IV the space-time symmetry of this collision integral and link the correlated density with the collision delay. In Section V we summarize and refer to numerical solutions of the derived nonlocal kinetic equation. The most important formulae for the T-matrix are presented in Appendix A and the technique of how to derive nonlocal corrections from gradient expansion is summarized in Appendix B.

2. KINETIC EQUATIONS

First we want to show that the Levinson equation can be transformed into the kinetic equation of Landau–Silin type and vice versa. Therefore we first derive the kinetic equation for the Wigner distribution from the real-time Green function technique. For a homogeneous system similar investigations have been presented by Bornath *et al.* [53]. Here we extend their approach to inhomogeneous systems and want to focus especially on the physical contents of both equations. This approach will furnish us with a set of Kramers–Kronig relations needed to link the Wigner and the quasiparticle distributions.

We consider the two independent correlation functions for Fermionic operators $G^>(1, 2) = \langle a(1)a^+(2) \rangle$ and $G^<(1, 2) = \langle a^+(2)a(1) \rangle$, where cumulative variables mean time and space, $1 \equiv t, x$. The time diagonal part of $G^<$ yields the Wigner distribution, $\rho(x_1, x_2, t) = G^<(1, 2)_{t_1, 2=t}$. For the correlation function $G^<$ the Kadanoff and Baym equation of motion reads [54]

$$-i(G_0^{-1}G^< - G^<G_0^{-1}) = -i(\Sigma^R G^< - G^<\Sigma^A) + i(G^R \Sigma^< - \Sigma^<G^A). \quad (7)$$

The retarded and advanced functions are introduced as $B^R(1, 2) = -i\theta(t_1 - t_2)(B^> + B^<)$ and $B^A(1, 2) = i\theta(t_2 - t_1)(B^> + B^<)$. The products are understood as integrations over intermediate variables (time and space). The Hartree–Fock drift term has the form

$$G_0^{-1}(1, 2) = \left(i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} \right) \delta(1 - 2) - \Sigma^{\text{hf}}(x_1, x_2, t_1) \delta(t_1 - t_2). \quad (8)$$

The Kadanoff and Baym equation (7) is more general than any of the above kinetic equations which can be derived from it with the help of specific approximations. One possibility is to rearrange the terms such that they yield the Landau–Silin kinetic equation for the quasiparticle distribution function (45) when the gradient expansion is applied [50, 55, 56]. The other possibility is to collect the terms on the right hand side of (7) into a non-Markovian collision integral which results in the Levinson equation (3) for the Wigner distribution [57–59]. The connection between both equations, however, has remained dubious because in the Levinson equation the drift term is controlled exclusively by the Hartree–Fock selfenergy while in the Landau–Silin equation the drift term is represented by the full quasiparticle energy. In the following we demonstrate how the Levinson equation can be converted into the Landau–Silin equation. To this end we first derive the Levinson equation from (7).

2.1. Precursor of the Levinson Equation

On the time diagonal, $t_{1,2} = t$, the Kadanoff and Baym equation (7) yields an identity

$$\begin{aligned}
 & -i(G_0^{-1}G^< - G^<G_0^{-1})(t, x_1, t, x_2) \\
 &= \int_{-\infty}^t dt' \int dx' (G^>(t, x_1, t', x')\Sigma^<(t', x', t, x_2) + \Sigma^<(t, x_1, t', x')G^>(t', x', t, x_2) \\
 & \quad - (G^<(t, x_1, t', x')\Sigma^>(t', x', t, x_2) + \Sigma^>(t, x_1, t', x')G^<(t', x', t, x_2)). \tag{9}
 \end{aligned}$$

On the left hand side there is the Hartree–Fock drift and the right hand side contains a non-Markovian collision integral. Note that correlations beyond the Hartree–Fock field are exclusively in the collision integral.

2.1.1. The Wigner Representation

Now we concentrate on the gradient expansion of the collision integral. The quasi-classical approximation is achieved using the Wigner mixed representation for space arguments

$$\hat{\sigma}\left(k, \frac{x_1 + x_2}{2}, \frac{t_1 + t_2}{2}, t_1 - t_2\right) = \int d(x_1 - x_2) e^{-ik(x_1 - x_2)} \Sigma(t_1, x_1, t_2, x_2). \tag{10}$$

As a rule, we use the lower case to denote operators in the full Wigner representation (momentum, space, time, frequency) and the hat to denote the Wigner representation in space and double-time representation. Note that time arguments of $\hat{\sigma}$ now denote the center-of-mass time, $(t_1 + t_2)/2$, and the difference time, $t_1 - t_2$. These two Wigner representations are identical for ρ and σ^{hf} which do not depend on the difference time.

In the representation (10) the identity (9) reads

$$\begin{aligned}
 & \left(\frac{\partial}{\partial t} + \frac{k}{m} \frac{\partial}{\partial r}\right) \rho(k, r, t) + 2 \sin\left(\frac{1}{2}(\partial_r^1 \partial_k^2 - \partial_k^1 \partial_r^2)\right) \sigma^{\text{hf}}(k, r, t) \rho(k, r, t) \\
 &= \exp\left(\frac{i}{2}(\partial_r^1 \partial_k^2 - \partial_k^1 \partial_r^2)\right) \int_0^\infty d\tau \left(\hat{g}^>\left(k, r, t - \frac{\tau}{2}, \tau\right) \hat{\sigma}^<\left(k, r, t - \frac{\tau}{2}, -\tau\right) \right. \\
 & \quad + \hat{\sigma}^<\left(k, r, t - \frac{\tau}{2}, \tau\right) \hat{g}^>\left(k, r, t - \frac{\tau}{2}, -\tau\right) - \hat{g}^<\left(k, r, t - \frac{\tau}{2}, \tau\right) \hat{\sigma}^>\left(k, r, t - \frac{\tau}{2}, -\tau\right) \\
 & \quad \left. - \hat{\sigma}^>\left(k, r, t - \frac{\tau}{2}, \tau\right) \hat{g}^<\left(k, r, t - \frac{\tau}{2}, -\tau\right)\right). \tag{11}
 \end{aligned}$$

Here, the superscripts 1, 2 denote that the partial derivatives apply only to the first and second function in the product, e.g., $(\partial_r^1 \partial_k^2)^3 ab = (\partial^3 a / \partial r^3) (\partial^3 b / \partial k^3)$.

2.1.2. Gradient Approximation

The power expansion of the goniometric functions in (11) defines the expansion in space gradients. This expansion goes to infinite order, but in the following we will restrict our attention to the linear

expansion. It is customary to abbreviate the linear gradient terms with the help of the Poisson brackets,

$$2 \sin\left(\frac{1}{2}(\partial_r^1 \partial_k^2 - \partial_k^1 \partial_r^2)\right) ab \approx \{a, b\} = \frac{\partial a}{\partial k} \frac{\partial b}{\partial r} - \frac{\partial a}{\partial r} \frac{\partial b}{\partial k}. \quad (12)$$

The linear approximation in space gradients is straightforward; however, one has to be careful about time arguments of functions in the collision integral of (11). For example, the first and second products seem to form an anticommutator in which the linear gradients in space cancel. This is not true because of time arguments.

The gradient approximation in time requires a special treatment because of the lower integration limit. Due to this limit the integral does not define a standard matrix product with respect to the time variables. We will give this expansion an explicit notation.

In equilibrium, the functions \hat{g}^{\approx} and $\hat{\sigma}^{\approx}$ do not depend on the center-of-mass time. For slowly evolving systems the center-of-mass dependence is smooth and weak. Accordingly, in the collision integral of (11), we can expand the center-of-mass time dependence around the time t in powers of τ . Since all functions in (11) have the same center-of-mass time, $t - \frac{\tau}{2}$, it is possible to write the linear expansion of (11) with respect to time gradients in a compact form

$$\frac{\partial}{\partial t} \rho + \{\epsilon^{\text{hf}}, \rho\} = I + \frac{\partial}{\partial t} R, \quad (13)$$

with $\epsilon^{\text{hf}} = k^2/2m + \sigma^{\text{hf}}$. The zero order gradient term, $I = I^> - I^<$, reads

$$I^< = \left(1 + \frac{i}{2}(\partial_r^1 \partial_k^2 - \partial_k^1 \partial_r^2)\right) \int_0^\infty d\tau (\hat{\sigma}^>(t, \tau) \hat{g}^<(t, -\tau) + \hat{g}^<(t, \tau) \hat{\sigma}^>(t, -\tau)). \quad (14)$$

$I^>$ is obtained from $I^<$ via the interchange of particles and holes, $>\leftrightarrow<$. In the first order gradient term, $R = R^> - R^<$, is given by

$$R^< = -\frac{1}{2} \left(1 + \frac{i}{2}(\partial_r^1 \partial_k^2 - \partial_k^1 \partial_r^2)\right) \int_0^\infty d\tau \tau (\hat{\sigma}^>(t, \tau) \hat{g}^<(t, -\tau) + \hat{g}^<(t, \tau) \hat{\sigma}^>(t, -\tau)). \quad (15)$$

Now we are ready to turn all functions into the Wigner representation for time and energy,

$$\sigma_\omega(k, r, t) = \int d\tau e^{i\omega\tau} \hat{\sigma}(k, r, t, \tau). \quad (16)$$

We will write the energy argument as the subscript and keep other arguments implicit in most of the formulas.

After a substitution of σ 's and g 's of the Wigner representation (16) into (14), one can integrate out the difference time τ . The integration over time results in the δ function which describes processes on the energy shell, and the principle value terms which represent off-shell processes,

$$\int_0^\infty d\tau e^{i(\omega-\omega')\tau} = \pi \delta(\omega - \omega') + i \frac{\wp}{\omega - \omega'}. \quad (17)$$

The δ -function can be readily integrated out leaving both functions with identical energy arguments. The principle value part, \wp , is an odd function of energies ω and ω' ; therefore its nongradient

contributions cancel but the linear gradients survive. Formula (14) thus turns into

$$I^< = \int \frac{d\omega}{2\pi} \sigma_{\omega}^> g_{\omega}^< + \int \frac{d\omega d\omega'}{(2\pi)^2} \frac{\wp}{\omega' - \omega} \{\sigma_{\omega'}^>, g_{\omega}^<\}. \quad (18)$$

In the correlated part we express the time integral as

$$\int_0^{\infty} d\tau \tau e^{i(\omega - \omega')\tau} = -i \frac{\partial}{\partial \omega} \int_0^{\infty} d\tau e^{i(\omega - \omega')\tau} = -i\pi \delta'(\omega - \omega') + \frac{\wp'}{\omega - \omega'}. \quad (19)$$

The meaning of δ' and \wp' is seen from comparison with (17). The δ' and \wp' are odd and even functions in $\omega - \omega'$, respectively. The gradient expansion of (15) thus reads

$$R^< = - \int \frac{d\omega d\omega'}{(2\pi)^2} \frac{\wp'}{\omega' - \omega} \sigma_{\omega'}^> g_{\omega}^< + \int \frac{d\omega}{2\pi} \{\sigma_{\omega}^>, \partial_{\omega} g_{\omega}^<\}. \quad (20)$$

The term R contributes to the kinetic equation (13) as a gradient correction linear in time. Since the second term of $R^<$ in (20) is proportional to space gradients, we neglect this term leading to the second order contribution in gradients.

2.2. Connection between the Wigner and Quasiparticle Distributions

At very low temperatures, dissipative processes are known to vanish due to the Pauli exclusion principle. It is desirable to reorganize the kinetic equation so that the scattering integral will vanish in this limit too. As can be seen from (20), the term $\partial_t R$ remains finite even for very low temperatures. It is possible to formally remove $\partial_t R$ if we shift this term on the drift side of (13) and introduce a new distribution function,

$$f = \rho - R. \quad (21)$$

We have denoted the new function as f to anticipate that it is indeed the quasiparticle distribution as will be shown now.

Since we want to arrive at the kinetic equation for f , it is advantageous to write relation (21) in the opposite way so that we obtain ρ as a functional of f , $\rho = f + R$. Using (20) and the particle-hole conjugated term, $R^>$, one finds

$$\rho = f - \int \frac{d\omega' d\omega}{(2\pi)^2} \frac{\wp'}{\omega' - \omega} (\sigma_{\omega'}^> g_{\omega}^< - \sigma_{\omega'}^< g_{\omega}^>). \quad (22)$$

This relation provides the so-called extended quasiparticle approximation of the Wigner distribution.

2.2.1. Wave-function Renormalization

To separate the on-shell and off-shell contributions of the correlation term, we use the imaginary part of the selfenergy, $\gamma = \sigma^> + \sigma^<$, and the spectral function, $a = g^> + g^<$, to rewrite (22) as

$$\rho = f - \int \frac{d\omega' d\omega}{(2\pi)^2} \frac{\wp'}{\omega' - \omega} (\gamma_{\omega'} g_{\omega}^< - \sigma_{\omega'}^< a_{\omega}). \quad (23)$$

With the help of the Kramers–Kronig relation

$$\int \frac{d\omega'}{2\pi} \frac{\wp'}{\omega - \omega'} \gamma_{\omega'} = \frac{\partial \sigma_{\omega}}{\partial \omega}, \quad (24)$$

where σ is the real part of the selfenergy, we obtain

$$\rho = f + \int \frac{d\omega}{2\pi} \frac{\partial \sigma_{\omega}}{\partial \omega} g_{\omega}^{\leq} + \int \frac{d\omega' d\omega}{(2\pi)^2} \frac{\wp'}{\omega' - \omega} \sigma_{\omega'}^{\leq} a_{\omega}. \quad (25)$$

The simplest quasiparticle approximation, $a_{\omega} = 2\pi \delta(\omega - \varepsilon)$ and $g_{\omega}^{\leq} = f 2\pi \delta(\omega - \varepsilon)$, in the correlation terms yields

$$\rho = \left(1 + \frac{\partial \sigma}{\partial \omega} \Big|_{\omega=\varepsilon} \right) f + \int \frac{d\omega'}{2\pi} \frac{\wp'}{\omega' - \varepsilon} \sigma_{\omega'}^{\leq}. \quad (26)$$

In (26) the on-shell (quasiparticle) contribution is reduced by the wave-function renormalization,

$$z = 1 + \frac{\partial \sigma}{\partial \omega} \Big|_{\omega=\varepsilon}, \quad (27)$$

and the term with σ^{\leq} provides the off-shell contributions.

The exact renormalization is $z^{-1} = 1 - \partial_{\omega} \sigma$; therefore (27) is only the linear approximation in $\partial_{\omega} \sigma$. From the Kramers–Kronig relation (24) one can see that this corresponds to the linear approximation in the scattering rate γ . We will keep all terms only up to this order in γ .

2.2.2. Extended Quasiparticle Approximation

From the relation between the Wigner and the quasiparticle distributions (26) one can deduce an approximative construction of the correlation function g^{\leq} as a functional of the quasiparticle distribution f . Since $\rho = \int \frac{d\omega}{2\pi} g^{\leq}$, we can write (26) as

$$\int \frac{d\omega}{2\pi} g_{\omega}^{\leq} = \int \frac{d\omega}{2\pi} \left(f z 2\pi \delta(\omega - \varepsilon) + \frac{\wp'}{\omega - \varepsilon} \sigma_{\omega}^{\leq} \right). \quad (28)$$

We have multiplied f with the δ function and included this term into the integrand.

One can see that the correlation function in the extended quasiparticle approximation [50, 56],

$$g_{\omega}^{\leq} = f z 2\pi \delta(\omega - \varepsilon) + \frac{\wp'}{\omega - \varepsilon} \sigma_{\omega}^{\leq}, \quad (29)$$

satisfies (28). Perhaps it is not necessary to recall that the function f has been introduced here merely to suppress the nondissipative part of the collision integral, $\partial_t R$. In [50, 56], the function f has been defined as the factor of the singularity of g^{\leq} which justifies its interpretation in terms of the quasiparticle distribution. As relation (28) shows, both approaches lead to identical results.

In parallel with (29), for the hole correlation function one finds

$$g_{\omega}^{\geq} = (1 - f) z 2\pi \delta(\omega - \varepsilon) + \frac{\wp'}{\omega - \varepsilon} \sigma_{\omega}^{\geq}. \quad (30)$$

From the spectral identity, $a = g^> + g^<$, it follows that (29) and (30) are consistent with the extended quasiparticle approximation of the spectral function

$$a_\omega = z2\pi\delta(\omega - \varepsilon) + \frac{\wp'}{\omega - \varepsilon}\gamma_\omega. \quad (31)$$

It is worthy to remark that (31) fulfills the spectral sum rule [47]

$$\int \frac{d\omega}{2\pi} a(k, \omega, r, t) = 1 \quad (32)$$

as well as the energy weighted sum rule [52]

$$\int \frac{d\omega}{2\pi} \omega a(k, \omega, r, t) = \frac{k^2}{2m} + \sigma^{\text{hf}}(k, r, t). \quad (33)$$

The latter one is violated in the simple quasiparticle picture.

2.3. Landau–Silin Equation

So far we have treated only the time gradients finding that the identity (13) can be expressed as

$$\frac{\partial f}{\partial t} + \{\varepsilon^{\text{hf}}, \rho\} = I. \quad (34)$$

Now we rearrange this identity into the Landau–Silin equation for f .

The time-local remainder I of the collision integral still includes the space gradients, $I = I_B - I_\nabla$. The nongradient part (the first term of (18)) is the Boltzmann-type scattering integral of the Landau–Silin equation

$$I_B = \int \frac{d\omega}{2\pi} (g_\omega^>\sigma_\omega^< - g_\omega^<\sigma_\omega^>) = z((1 - f)\sigma_\varepsilon^< - f\sigma_\varepsilon^>). \quad (35)$$

In the second step we have used (29). The off-shell contributions to (35) have not been neglected but cancel exactly.

The gradient part which is the second term of (18) reads

$$I_\nabla = - \int \frac{d\omega d\omega'}{(2\pi)^2} \frac{\wp}{\omega' - \omega} (\{\sigma_{\omega'}^>, g_\omega^<\} - \{\sigma_{\omega'}^<, g_\omega^>\}) = \int \frac{d\omega}{(2\pi)} (\{\sigma_\omega, g_\omega^<\} - \{\sigma_\omega^<, g_\omega\}). \quad (36)$$

Here (24) has been used to remove the ω' -integration for $\sigma^> + \sigma^<$ and a similar Kramers–Kronig relation for the ω -integration over the spectral function $g^> + g^<$ has been applied. The function $g = \text{Re}g^{R,A} = \frac{1}{2}(g^R + g^A)$ is the off-shell part of the propagators.

The mean-field drift $\{\varepsilon^{\text{hf}}, \rho\}$ in (34) is not compatible with the quasiparticle distribution in the time derivative and the scattering integral. Moreover, the space gradients from the collision integral, I_∇ , have to be accounted for. Our aim is now to show that all gradients can be collected into the quasiparticle drift,

$$\{\varepsilon^{\text{hf}}, \rho\} + I_\nabla = \{\varepsilon, f\}. \quad (37)$$

The quasiparticle energy differs from the mean-field energy by the pole value of the selfenergy, $\varepsilon = \varepsilon^{\text{hf}} + \sigma_\varepsilon$.

We will proceed in two steps. First we observe that the right hand side of (37) includes only the on-shell contribution; therefore we show that the on-shell part of the left hand side equals the right hand side. Second, we show that the off-shell part of the left hand side of (37) vanishes. The on-shell parts of (29) used in (36) result in

$$\begin{aligned} I_{\nabla}^{\text{on}} &= \int d\omega \{ \sigma_{\omega}, z f \delta(\omega - \varepsilon) \} = \{ \sigma_{\varepsilon}, z f \} - \sigma'_{\varepsilon} \{ \varepsilon, z f \} + z f \{ \sigma'_{\varepsilon}, \varepsilon \} \\ &= \{ \sigma_{\varepsilon}, z f \} + (1 - z) \{ \varepsilon, z f \} + z f \{ z - 1, \varepsilon \}. \end{aligned} \quad (38)$$

Now we add the on-shell part of the commutator $\{ \epsilon^{\text{hf}}, \rho \}^{\text{on}} = \{ \epsilon^{\text{hf}}, z f \}$. Using a rearrangement

$$\{ \epsilon^{\text{hf}}, z f \} = \{ \varepsilon - \sigma, z f \} = \{ \varepsilon, f \} + (z - 1) \{ \varepsilon, f \} + f \{ \varepsilon, z \} - \{ \sigma, z f \} \quad (39)$$

we obtain

$$\{ \epsilon^{\text{hf}}, \rho \}^{\text{on}} + I_{\nabla}^{\text{on}} = \{ \varepsilon, f \} - \{ \varepsilon, (1 - z)^2 f \}. \quad (40)$$

The last term is of higher order than linear in the damping and can be neglected which confirms the relation (37) already from the on-shell parts.

It remains to prove that all off-shell contributions to the left hand side of (37), $\mathcal{O} = \{ \epsilon^{\text{hf}}, \rho \}^{\text{off}} + I_{\nabla}^{\text{off}}$ cancel, i.e., $\mathcal{O} = 0$. From the off-shell term of (29) we find

$$\mathcal{O} = \int \frac{d\omega}{2\pi} \left(\left\{ \epsilon^{\text{hf}} + \sigma_{\omega}, \frac{\wp'}{\omega - \varepsilon} \sigma_{\omega}^{\leftarrow} \right\} - \{ g_{\omega}, \sigma_{\omega}^{\leftarrow} \} \right). \quad (41)$$

The term with ϵ^{hf} results from $\{ \epsilon^{\text{hf}}, \rho \}$, while the others from I_{∇} . In the last term we use the extended quasiparticle approximation of the real part of the propagator

$$g_{\omega} = z \frac{\wp}{\omega - \varepsilon} + \frac{\wp'}{\omega - \varepsilon} (\sigma_{\omega} - \sigma_{\varepsilon}) \quad (42)$$

which directly results from (31) and the Kramers–Kronig relation. Using $\{ \frac{\wp}{\omega - \varepsilon}, \sigma_{\omega}^{\leftarrow} \} = -\{ \varepsilon, \frac{\wp'}{\omega - \varepsilon} \sigma_{\omega}^{\leftarrow} \}$ we can rewrite \mathcal{O} as

$$\mathcal{O} = \int \frac{d\omega}{2\pi} \left(\left\{ \sigma_{\omega} - \sigma_{\varepsilon}, \frac{\wp'}{\omega - \varepsilon} \sigma_{\omega}^{\leftarrow} \right\} - \left\{ \frac{\wp'}{\omega - \varepsilon} (\sigma_{\omega} - \sigma_{\varepsilon}), \sigma_{\omega}^{\leftarrow} \right\} \right). \quad (43)$$

The linear expansion in the vicinity of the pole, $\sigma_{\omega} - \sigma_{\varepsilon} = (\omega - \varepsilon)(z - 1) + o(\gamma^2)$, yields

$$\mathcal{O} = \int \frac{d\omega}{2\pi} \left\{ \frac{\wp''}{\omega - \varepsilon}, (z - 1) \sigma_{\omega}^{\leftarrow} \right\}. \quad (44)$$

The product $(z - 1) \sigma^{\leftarrow}$ is of second order in γ ; i.e., the off-shell contribution \mathcal{O} is negligible within the assumed accuracy. This completes the proof of relation (37).

In summary, the requirement that the scattering integral vanish at very low temperatures directly leads to the concept of quasiparticles represented by relation (22) between the Wigner and the quasiparticle distributions

$$\rho = f + \int \frac{d\omega}{2\pi} \frac{\wp}{\omega - \varepsilon} \frac{\partial}{\partial \omega} ((1 - f) \sigma_{\omega}^{\leftarrow} - f \sigma_{\omega}^{\rightarrow}). \quad (45)$$

The space gradients of the collision integral renormalize the mean-field drift into the familiar quasiparticle drift (37). The resulting kinetic equation for the quasiparticle distribution has the structure of the Landau–Silin equation (2)

$$\frac{\partial f}{\partial t} + \frac{\partial \varepsilon}{\partial k} \frac{\partial f}{\partial r} - \frac{\partial \varepsilon}{\partial r} \frac{\partial f}{\partial k} = z((1-f)\sigma_\varepsilon^< - f\sigma_\varepsilon^>). \quad (46)$$

We conclude this section with the Levinson type equation (3) being equivalent to the Landau–Silin equation (2) up to second order in the damping γ or in the extended quasiparticle picture.

3. ADDITIONAL RETARDATION OF THE LEVINSON EQUATION

The link between the Levinson and Landau–Silin equations shows that the collision integral of the Levinson equation cannot be fully interpreted in terms of the scattering processes but it includes various renormalization features. In the above treatment we have constructed a closed equation for the quasiparticle distribution function while the reduced density is given by an additional functional of this quasiparticle distribution. There arises an important question whether it is possible not to construct the correlation functions, $g^{>,<}$, directly from the Wigner distribution so that identity (11) turns into a closed equation for the Wigner distribution. To show the problems and drawbacks with this opposite way we discuss this construction for a homogeneous system.

An approximation giving the Levinson equation is the GKB ansatz [58] which for the homogeneous system reads [$\tau > 0$]

$$\begin{aligned} \hat{g}^<(k, t - \frac{\tau}{2}, \tau) &= i\hat{g}^R(k, t - \frac{\tau}{2}, \tau)\rho(k, t - \tau), \\ \hat{g}^<(k, t - \frac{\tau}{2}, -\tau) &= -i\hat{g}^A(k, t - \frac{\tau}{2}, -\tau)\rho(k, t - \tau). \end{aligned} \quad (47)$$

For $\hat{g}^>$ one substitutes $1 - \rho$ for ρ . Note that this approximation includes the additional retardation between the center-of-mass time $t - \frac{\tau}{2}$ and the time argument of the Wigner distribution which is $t - \tau$. This additional retardation causes many problems ranging from instability of numerical treatments over double counts of renormalizations or virial corrections to incorrect interpretations of various correlation phenomena. Here we show how to handle this retardation within the linear approximation.

If one assumes that the propagator G^R depends only on the difference time τ , i.e., it does not depend on any time dependent external fields or mean fields, it is possible to follow the approach developed above with only minor modifications. To make our discussion specific we limit our attention to homogeneous systems and employ the Galitskii–Feynman T-matrix approximation of the selfenergy,

$$\begin{aligned} \hat{\sigma}^<(k, t - \frac{\tau}{2}, \tau) &= \int \frac{dp dq}{(2\pi)^6} \hat{g}^>(p, t - \frac{\tau}{2}, -\tau) \int d\tau_R d\tau_A T^R(k, p, q, \tau_R) T^A(k, p, q, -\tau_A) \\ &\quad \times \hat{g}^<(k - q, t - \frac{1}{2}(\tau_R + \tau_A + \tau), \tau - \tau_R + \tau_A) \\ &\quad \times \hat{g}^<(p + q, t - \frac{1}{2}(\tau_R + \tau_A + \tau), \tau - \tau_R + \tau_A). \end{aligned} \quad (48)$$

This selfenergy describes the process in which two particles of initial momenta $k - q$ and $p + q$

are scattered into final states k and p . The function $\mathcal{T}^{R,A}$ is the retarded/advanced scattering T-matrix; see Appendix A. Due to a finite duration of the scattering process, the T-matrices bring yet additional retardation times $\tau_{R,A}$. We will show that retardation times $\tau_{R,A}$ are responsible for the virial corrections to the density.

3.1. Decay of Propagators and Double Counts

First we focus on double counts following from the additional retardation resulting from the GKB ansatz. Since these problems appear already for weakly coupled systems, it is sufficient to restrict our attention to the Born approximation, $\mathcal{T}^{R,A} = \mathcal{V}\delta(\tau_{R,A})$. The Levinson equation then reads

$$\frac{\partial}{\partial t}\rho(k, t) = 2\text{Re} \int \frac{dp dq}{(2\pi)^6} \mathcal{V}_q^2 \int_0^\infty d\tau S\left(t - \frac{\tau}{2}, \tau\right) D(t - \tau). \quad (49)$$

All distributions are collected in

$$D(t) = (1 - \rho(k, t))(1 - \rho(p, t))\rho(k - q, t)\rho(p + q, t) - \rho(k, t)\rho(p, t)(1 - \rho(k - q, t))(1 - \rho(p + q, t)); \quad (50)$$

all propagators are collected in

$$S(t, \tau) = \hat{g}^R(k, t, \tau)\hat{g}^R(p, t, \tau)\hat{g}^A(k - q, t, -\tau)\hat{g}^A(p + q, t, -\tau). \quad (51)$$

We have used that the Wigner distribution is a real function and the complex conjugacy of propagators, $\hat{g}^R(k, t, \tau) = (\hat{g}^A(k, t, -\tau))^*$, to reduce the expressions.

Let us assume for the moment that the propagators do not depend on the center-of-mass time t . In this case the only t dependence in the collision integral of the Levinson equation (49) is due to distributions. Since for the Born approximation all distributions are retarded in an equal way, we can follow a modification of the approach used above.

For slow perturbations one can expand (49) to the lowest order in the memory, $D(t - \tau) = D(t) - \tau \partial_t D(t)$, so that the collision integral splits into the Boltzmann-like scattering integral and the gradient correction, $\partial_t \rho = \mathcal{I} + \partial_t \mathcal{R}$, with

$$\mathcal{I} = \int \frac{dp dq}{(2\pi)^6} \mathcal{V}_q^2 D(t) 2 \text{Re} \int_0^\infty d\tau S(\tau), \quad (52)$$

$$\mathcal{R} = \int \frac{dp dq}{(2\pi)^6} \mathcal{V}_q^2 D(t) 2 \text{Re} \int_0^\infty d\tau \tau S(\tau). \quad (53)$$

Please note that \mathcal{R} and \mathcal{I} are different from (13). Again, from (49) we can define a new function to remove the retardation,

$$\mathcal{F} = \rho - \mathcal{R}, \quad (54)$$

and try to redefine the transport in terms of \mathcal{F} . This will lead us now to unavoidable double counts.

3.1.1. Double Counts

The integrals in (52) are controlled by two mechanisms, the de-phasing given by momentum integrals and the decay of propagators. Let us assume for a while that the de-phasing is the dominant part leaving the decay aside. Within the quasiparticle approximation, $\hat{g}^R(k, \tau) \approx -i e^{-i\varepsilon_k \tau}$, which is free of the decay, the time integrals over propagators yield

$$2 \operatorname{Re} \int_0^{\infty} d\tau S_{\text{qp}}(\tau) = 2\pi \delta(\varepsilon_k + \varepsilon_p - \varepsilon_{k-q} - \varepsilon_{p+q}), \quad (55)$$

$$2 \operatorname{Re} \int_0^{\infty} d\tau \tau S_{\text{qp}}(\tau) = -2 \frac{\wp'}{\varepsilon_k + \varepsilon_p - \varepsilon_{k-q} - \varepsilon_{p+q}}. \quad (56)$$

After approximation (55), \mathcal{I} from (52) turns into the Boltzmann-type scattering integral I_B of the Landau–Silin equation. Naturally, the on-shell contributions are identical since they do not depend on the retardation.

Using (56) in (53) one finds

$$\mathcal{R} = 2 \int \frac{dp dq}{(2\pi)^6} \mathcal{V}_q^2 D(t) \frac{\wp'}{\varepsilon_k + \varepsilon_p - \varepsilon_{k-q} - \varepsilon_{p+q}}. \quad (57)$$

The function \mathcal{R} is twice the off-shell contribution R to the Wigner distribution. To show this, we express (57) in terms of the Born selfenergy (48) and the approximative correlation functions, $g^< = 2\pi \delta(\omega - \varepsilon)\rho$, as

$$\mathcal{R} = -2 \int \frac{d\omega' d\omega}{(2\pi)^2} \frac{\wp'}{\omega - \omega'} (\sigma_{\omega'}^> g_{\omega}^< - \sigma_{\omega'}^< g_{\omega}^>). \quad (58)$$

Comparing with the second term of (22) one can see that the additional retardation results in the double count of the correlated part, $\mathcal{R} = 2R$. Due to this double count, the function $\mathcal{F} = \rho - \mathcal{R} = \rho - 2R = f - R$ cannot be interpreted as the distribution of excitations because it has large regions of negative values. Please note a common pitfall found in the literature related to this double count given in Appendix C.

We note that in numerical treatments of the Levinson equation neglecting or underestimating the decay leads to a numerical instability of the equation, as it has been reported by Haug [60]. Other numerical solutions show a continuous increase of kinetic energy, e.g., Fig. 4 of [61] or pulsation modes [62]. The instability of the solution is a problem which shows that the Levinson equation is a difficult tool to handle. The double count of the off-shell contribution resulting in the negative effective occupation factors is likely one of the reasons for this problem. A detailed discussion and numerical comparison of the Levinson equation with the Kadanoff and Baym equation can be found in [63].

3.1.2. Decay of Propagators

The double count of correlations is a serious mistake for theories which aim to go beyond the Boltzmann equation. Here we show that the decay of propagators during the collision removes this double count.

We use the simplest approximation of the decay,

$$\hat{g}^R(k, t, \tau) = -i e^{-i\varepsilon_k \tau} e^{-\gamma_k \frac{\tau}{2}}, \quad (59)$$

where $\varepsilon_k = \varepsilon(k, t - \frac{\tau}{2})$ and $\gamma_k = \gamma_{\varepsilon(k, t - \tau/2)}(k, t - \frac{\tau}{2})$ is the pole value of $\gamma = \sigma^> + \sigma^< = -2 \operatorname{Im} \sigma^R$. The time argument shows that external or mean fields can modify the propagation during the collision. The collected propagator thus reads

$$S\left(t - \frac{\tau}{2}, \tau\right) = S_{\text{qp}}\left(t - \frac{\tau}{2}, \tau\right) e^{-(\gamma_k + \gamma_p + \gamma_{k-q} + \gamma_{p+q})\frac{\tau}{2}}. \quad (60)$$

The kinetic equation holds only if the collision duration is short on the scale of the lifetime, $(2\gamma)^{-1}$. In this limit we can expand the exponential decay,

$$e^{-(\gamma_k + \gamma_p + \gamma_{k-q} + \gamma_{p+q})\frac{\tau}{2}} = 1 - (\gamma_k + \gamma_p + \gamma_{k-q} + \gamma_{p+q})\frac{\tau}{2}. \quad (61)$$

Since we are dealing with the correction terms, we can use a simple approximation of the kinetic equation, $\partial_t \rho_k = \sigma_k^< - \gamma_k \rho$, to rearrange the contribution of γ 's to its integrand,

$$-\frac{\tau}{2}(\gamma_k + \gamma_p + \gamma_{k-q} + \gamma_{p+q}) = \frac{\tau}{2} \frac{\partial}{\partial t} D - \frac{\tau}{2} D_{3p}. \quad (62)$$

The last term,

$$\begin{aligned} D_{3p} = & \sigma_k^> (1 - \rho_p) \rho_{k-q} \rho_{p+q} + (1 - \rho_k) \sigma_p^> \rho_{k-q} \rho_{p+q} + (1 - \rho_k)(1 - \rho_p) \sigma_{k-q}^< \rho_{p+q} \\ & + (1 - \rho_k)(1 - \rho_p) \rho_{k-q} \sigma_{p+q}^< - \sigma_k^< \rho_p (1 - \rho_{k-q})(1 - \rho_{p+q}) - \rho_k \sigma_p^< \\ & \times (1 - \rho_{k-q})(1 - \rho_{p+q}) - \rho_k \rho_p \sigma_{k-q}^> (1 - \rho_{p+q}) - \rho_k \rho_p (1 - \rho_{k-q}) \sigma_{p+q}^>, \end{aligned} \quad (63)$$

can be neglected leading to the three-particle contribution to the scattering integral. Please note that such neglects have not been necessary in our suggested construction $\rho[f]$ outlined in Section II.

Within the linear approximation in time gradients one finds that the decay reduces the retardation,

$$e^{-(\gamma_k + \gamma_p + \gamma_{k-q} + \gamma_{p+q})\frac{\tau}{2}} D(t - \tau) = D\left(t - \frac{\tau}{2}\right). \quad (64)$$

Now the collective propagator and distribution have the same time retardation of the center-of-mass time being equal to $\frac{\tau}{2}$. This time shift removes the double count and reproduces the result obtained already in Section II.

We note that the original Levinson equation [42] does not include the decay of propagators in the scattering integral. The presented discussion shows that the decay plays the important role for the correct treatment of renormalizations and other contributions of correlations.

3.2. Collision Delay

We have seen that within the Born approximation the retarded collision integral of the Levinson equation can be transformed into the instantaneous scattering integral of the Boltzmann type and quasiparticle renormalizations of the single-particle distribution and energy. Beyond the Born approximation, there is the additional retardation expressed by times $\tau_{R,A}$ in the selfenergy (48). This retardation is responsible for the virial correction, in particular for the correlated density n_c . Since the retardation times $\tau_{R,A}$ do not enter the final states of the collision, they cannot be removed in the above manner. On the other hand, the time integrals over $\tau_{R,A}$ can be eliminated so that the collision duration is expressed by an effective time-nonlocality of the scattering integral. For simplicity

of presentation in this section we assume that T-matrices and selfenergies are independent of the center-of-mass time. This will lead us to the basic idea. The complete result with all time and space dependences is given in the next Section IV.

Within the linear approximation in time gradients we can treat the contributions of times $\tau_{R,A}$ separately from other retardations. We first implement the quasiparticle ansatz and expand in retardations,

$$\begin{aligned} \hat{g}^<(k-q, t - \frac{1}{2}(\tau_R + \tau_A + \tau), \tau - \tau_R + \tau_A) \hat{g}^<(p+q, t - \frac{1}{2}(\tau_R + \tau_A + \tau), \tau - \tau_R + \tau_A) \\ = \left(1 - \frac{1}{2}\tau \frac{\partial}{\partial t} - \frac{1}{2}(\tau_R + \tau_A) \frac{\partial}{\partial t}\right) f_{k-q}(t) f_{p+q}(t) e^{-i(\varepsilon_{k-q} + \varepsilon_{p+q})(\tau - \tau_R + \tau_A)}. \end{aligned} \quad (65)$$

The contribution of $\frac{1}{2}\tau \frac{\partial}{\partial t}$ has been discussed above; now we focus on the term due to $\frac{1}{2}(\tau_R + \tau_A) \frac{\partial}{\partial t}$.

Introducing the energy representation of the T-matrix,

$$\mathcal{T}^R(\tau_R) = \int \frac{d\omega_R}{2\pi} t^R(\omega_R) e^{-i\omega_R \tau_R}, \quad (66)$$

and a complex conjugate for \mathcal{T}^A , the contribution of $\frac{1}{2}(\tau_R + \tau_A) \frac{\partial}{\partial t}$ to the selfenergy (48) reads

$$\begin{aligned} \Delta \hat{\sigma}^<(k, t, \tau) = -\frac{1}{2} \int \frac{dp dq}{(2\pi)^6} e^{i\varepsilon_p \tau} (1 - f_p) \int \frac{d\omega_R d\omega_A}{(2\pi)^2} t^R(k, p, q, \omega_R) t^A(k, p, q, \omega_A) \\ \times \int d\tau_R d\tau_A e^{-i\omega_R \tau_R} e^{i\omega_A \tau_A} (\tau_R + \tau_A) \frac{\partial}{\partial t} f_{k-q} f_{p+q} e^{-i(\varepsilon_{k-q} + \varepsilon_{p+q})(\tau - \tau_R + \tau_A)}. \end{aligned} \quad (67)$$

The integral over $\tau_{R,A}$ results in the derived δ -functions which can be readily integrated out,

$$\Delta \hat{\sigma}^<(k, t, \tau) = \int \frac{dp dq}{(2\pi)^6} e^{i(\varepsilon_p - \varepsilon_{k-q} - \varepsilon_{p+q})\tau} (1 - f_p) \frac{1}{2i} \left(\frac{\partial t^R}{\partial \omega} t^A - t^R \frac{\partial t^A}{\partial \omega} \right)_{\omega = \varepsilon_{k-q} + \varepsilon_{p+q}} \frac{\partial}{\partial t} f_{k-q} f_{p+q}. \quad (68)$$

The correction (68) is proportional to the collision delay defined from the phase shift,

$$\Delta_t = \frac{\partial \phi}{\partial \omega}. \quad (69)$$

Indeed, writing the T-matrices in the form $t^{R,A} = |t|^{\mp i\phi}$, where $|t|^2$ is the amplitude of the scattering rate and ϕ is the scattering phase shift, we obtain

$$\frac{1}{2i} \left(\frac{\partial t^R}{\partial \omega} t^A - t^R \frac{\partial t^A}{\partial \omega} \right) = -|t|^2 \Delta_t. \quad (70)$$

This allows us to join the correction with the time-local part of the selfenergy,

$$\hat{\sigma}^<(k, t, \tau) = \int \frac{dp dq}{(2\pi)^6} |t|^2 e^{i(\varepsilon_p - \varepsilon_{k-q} - \varepsilon_{p+q})\tau} (1 - f_p(t)) f_{k-q}(t - \Delta_t) f_{p+q}(t - \Delta_t). \quad (71)$$

The selfenergy (71) can be compared with (48). First, the retardation by $-\tau/2$ has been suppressed because it is eliminated by the method described in Section II. Second, the retardations by the internal times of the T-matrices, $\tau_{R,A}$, have been reduced from the integral form of (48) to the effective time nonlocality of (71). The latter form includes only a single characteristic time Δ_t which is similar (but not fully identical) to the collision delay defined by Wigner from the scattering phase shift. Note that in form (71) no time integration appears.

An interpretation of the collision delay Δ_t is troubled by the fact that this effective time does not have to be positive. Apparently, one cannot interpret the collision delay as a mean time scale of the $\tau_{R,A}$ integrals in the selfenergy (48). The negative value of Δ_t appears when the phase shift decreases with increasing energy, as it happens above a bound state or a resonant scattering state. For instance, the collision delay of nucleon–nucleon collisions is negative at low relative energies due to the deuteron bound state; see [64]. The negative value of the collision delay merely tells us that particles anti-correlate at a given relative energy.

4. NONLOCAL SCATTERING INTEGRAL

The selfenergy (71) is easily converted into the scattering integral. The $|t|^2$ represents the scattering amplitude in the T-matrix approximation. The exponential function turns into the energy conserving δ -function. Compared to the common scattering integral of the Landau–Silin equation, the only difference consists in the collision delay Δ_t entering the initial states of the collision. Of course, in the previous section we have assumed a homogeneous system with time-independent selfenergies and T-matrices. For a general case more nonlocal corrections appear. These corrections we discuss in this section.

In this section we thus focus on the scattering integral. Our aim is to derive the scattering integral within the same approximations as the skeleton kinetic equation discussed in Section II, i.e., keeping all linear gradients. This procedure leads to the noninstantaneous and nonlocal scattering integral. We use the Galitskii–Feynman T-matrix approximation of the selfenergy which is appropriate for dense systems of particles interacting via short range potentials. The T-matrix approximation for nonequilibrium Green functions is briefly presented in Appendix A.

4.1. Quasi-classical Limit

The consistent treatment of gradient contributions is provided by the quasi-classical limit, i.e., the linear expansion in gradients. The collision delay derived above shows that beside gradient contributions forming the drift in the skeleton equation, there are nontrivial gradient contributions making the scattering integral nonlocal in time. Transforming the system into a running coordinate framework, one can see that the collision delay has to be accompanied by the space nonlocality of the collision integral. Accordingly, the gradient expansion has to be applied also to all internal space and time integrations of the scattering integral.

The quasi-classical limit and the limit of small scattering rates explicitly determine how to evaluate the scattering integral from the selfenergy $\sigma^<$. For nondegenerate systems, a very similar scheme was carried out by Bärwinkel [1, 65]. One can see in Bärwinkel’s papers that the scattering integral is troubled by a large set of gradient corrections. This formal complexity seems to be the main reason why most authors either neglect gradient corrections completely [55, 66] or provide them buried in multi-dimensional integrals [30, 31, 67]. For a degenerate system, the set of gradient corrections to the scattering integral is even larger than for rare gases studied by Bärwinkel; see [51]. To avoid manipulations with long and obscure formulas, the gradient corrections have to be sorted and expressed in a comprehensive form. To this end, we will express all gradient corrections in the form of shifted arguments, as we have done above for the collision delay.

The quasi-classical limit of the scattering integral with all linear gradients retained is a tedious but straightforward algebraic exercise, see Appendix B, giving

$$\begin{aligned}
 \sigma^<(\omega, k, r, t) &= \int \frac{dp}{(2\pi)^3} \frac{dE}{2\pi} \frac{dq}{(2\pi)^3} \frac{d\Omega}{2\pi} \left(1 - \frac{1}{2} \frac{\partial \Delta_2}{\partial r} \right) \\
 &\times \left| t \left(\omega + E - \Delta_E, k - \frac{1}{2} \Delta_K, p - \frac{1}{2} \Delta_K, q, r - \Delta_r, t - \frac{1}{2} \Delta_t \right) \right|^2 \\
 &\times g^>(E, p, r - \Delta_2, t) g^<(\omega - \Omega - \Delta_E, k - q - \Delta_K, r - \Delta_3, t - \Delta_t) \\
 &\times g^<(E + \Omega - \Delta_E, p + q - \Delta_K, r - \Delta_4, t - \Delta_t). \tag{72}
 \end{aligned}$$

With $g^{>,<}$ substituted from the quasiparticle approximation, this formula turns into the functional of the quasiparticle distribution. As one can see, the initial states, $k - q$ and $p + q$, include the collision delay Δ_t , as in the homogeneous system describe by (71).

In addition to the collision delay, non-local selfenergy (72) includes a number of other gradient corrections which enter nearly all arguments. These are expressed via Δ 's which are derivatives of the scattering phase shift ϕ , where $t^{R,A} = |t|e^{\mp i\phi}$, according to the following list [51, 52]

$$\Delta_2 = \frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial k}, \tag{73}$$

$$\Delta_3 = -\frac{\partial \phi}{\partial k}, \tag{74}$$

$$\Delta_4 = -\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial k}, \tag{75}$$

$$\Delta_t = \frac{\partial \phi}{\partial \omega}, \tag{76}$$

$$\Delta_K = \frac{1}{2} \frac{\partial \phi}{\partial r}, \tag{77}$$

$$\Delta_E = -\frac{1}{2} \frac{\partial \phi}{\partial t}, \tag{78}$$

$$\Delta_r = \frac{1}{4} (\Delta_2 + \Delta_3 + \Delta_4). \tag{79}$$

Finally, there is the energy gain which is discussed in [68]. Numerical values of the delay Δ_t and the displacements Δ_{2-4} for the scattering of isolated particles are presented in [64].

Sending all Δ 's to zero, one recovers the instantaneous and local approximation of the selfenergy, Eq. (72). The noninstantaneous and nonlocal corrections given by Δ 's appear in three ways: in arguments of the correlation functions, in arguments of the T-matrix, and as a fore-factor. In correlation functions, the Δ 's describe displacements of the initial and final positions of colliding particles; the final state of one of the particles is fixed to the balanced phase-space point (k, r, t) . The arguments of the T-matrix correspond to the center of arguments of all initial and final states.

The pre-factor has no special physical meaning as it depends on the actual choice of energy and momentum variables. For instance, using in (72) an integral over $E' = E - \Delta_E$ instead of the integral over E , a corresponding factor, $dE/dE' = 1 + \partial_\omega \Delta_E = 1 - \frac{1}{2} \partial_t \Delta_t$, appears.

4.2. Extended Quasiparticle Approximation

Now we can complete the derivation of the kinetic equation. Since the scattering integral is already proportional to the “small” scattering rate, we can use from the extended quasiparticle approximation (29) the pole part, $g^<(\omega, k, r, t) = f(k, r, t)2\pi\delta(\omega - \varepsilon(k, r, t))$, to convert the selfenergies $\sigma^{> \cdot <}$ into functionals of the quasiparticle distribution. The selfenergy $\sigma^<$ is given by (72) and $\sigma^>$ is obtained by the interchange $> \leftrightarrow <$. The resulting kinetic equation reads

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \frac{\partial \varepsilon_1}{\partial k} \frac{\partial f_1}{\partial r} - \frac{\partial \varepsilon_1}{\partial r} \frac{\partial f_1}{\partial k} &= \int \frac{dp dq}{(2\pi)^6} 2\pi\delta(\varepsilon_1 + \bar{\varepsilon}_2 - \bar{\varepsilon}_3 - \bar{\varepsilon}_4 - 2\Delta_E) \left(1 - \frac{1}{2} \frac{\partial \Delta_2}{\partial r} - \frac{\partial \bar{\varepsilon}_2}{\partial r} \frac{\partial \Delta_2}{\partial \omega} \right) \\ &\times \left| t \left(\varepsilon_1 + \bar{\varepsilon}_2 - \Delta_E, k - \frac{1}{2} \Delta_K, p - \frac{1}{2} \Delta_K, q, r - \Delta_r, t - \frac{1}{2} \Delta_t \right) \right|^2 \\ &\times ((1 - f_1)(1 - \bar{f}_2)\bar{f}_3\bar{f}_4 - f_1\bar{f}_2(1 - \bar{f}_3)(1 - \bar{f}_4)). \end{aligned} \quad (80)$$

The abbreviated notation of f 's and ε 's means

$$\begin{aligned} \varepsilon_1 &\equiv \varepsilon(k, r, t), \\ \bar{\varepsilon}_2 &\equiv \varepsilon(p, r - \Delta_2, t), \\ \bar{\varepsilon}_3 &\equiv \varepsilon(k - q - \Delta_K, r - \Delta_3, t - \Delta_t), \\ \bar{\varepsilon}_4 &\equiv \varepsilon(p + q - \Delta_K, r - \Delta_4, t - \Delta_t). \end{aligned} \quad (81)$$

The bar reminds that all Δ 's appear in arguments with negative signs. In the Δ 's and their derivatives the energy $\omega + E = \varepsilon_1 + \bar{\varepsilon}_2$ is substituted after all derivatives are taken. Note that the pre-factor has changed compared to (72). In (72), $\bar{\varepsilon}_2$ depends on Δ_2 which depends on the energy E . This E -dependence has resulted in a norm of the singularity.

The nonlocal kinetic equation (80) represents the main result of this paper. In the following we will discuss some symmetry properties of this equation.

4.3. Markovian Approximation

In the scattering integrals of (81) the collision delay Δ_t appears together with displacements Δ_{2-4} . Actual values of these types of corrections are closely linked. In the original Wigner's approach, the collision delay was identified from the position of the center of wave packets in the final state. Apparently, this position can be described either as the space displacement or via the delay. A corresponding rearrangement of the scattering integral is achieved using the free-space time evolution of the distribution, e.g.,

$$\bar{f}_3 = f(k - q - \Delta_K, r - \Delta_3, t - \Delta_t) \approx f(k - q - \Delta_K + F_3\Delta_t, r - \Delta_3 + v_3\Delta_t, t), \quad (82)$$

where $v_3 = \partial_k \varepsilon_3$ is a velocity and $F_3 = -\partial_r \varepsilon_3$ is a force acting on the particle in the final state. Substituting the right hand side (and similar for \bar{f}_4 and $\bar{\varepsilon}_{3,4}$) into (81), one obtains instantaneous scattering integrals. This liberty of rearrangement is only approximative; some quantities are sensitive to such approximations. For instance, the correlated density discussed below is directly proportional to Δ_t as it measures an effective number of particles in the collision process. On the other hand, the rearrangement into the instantaneous form makes the kinetic equation Markovian what significantly simplifies its numerical treatment, in particular if negative collision delay takes place.

4.4. Space-Time Symmetry of Nonlocal Corrections

The scattering-out integral in (80) has the same sign of Δ 's as the scattering-in. Accordingly, due to the nonlocal and noninstantaneous corrections, the scattering-out cannot be obtained from the scattering-in by the space-time symmetry, as it is common in the classical approach to the kinetic equation. This problem has a practical impact. The simulations of pressure with noninstant corrections has been discussed by [69]. In Monte-Carlo simulations based on semi-classical trajectories the space-time symmetry is an unavoidable part of the scattering integral. To understand internal times of collisions and nonlocal corrections, it is thus necessary to clarify the space-time symmetry.

It is possible to show that the scattering-in and -out are connected by the space-time symmetry; one has to take into account, however, the nonlocal and noninstantaneous corrections to the T-matrix itself. To this end we employ the optical theorem which can be expressed in two forms,

$$\text{Im } \mathcal{T} = \mathcal{T}^R \mathcal{A} \mathcal{T}^A = \mathcal{T}^A \mathcal{A} \mathcal{T}^R. \quad (83)$$

Here, symbol Im denotes the anti-Hermitian part. The multiplication includes integrals over two-particle states and times. The function $\mathcal{A}_{34} = G_3^> G_4^> - G_3^< G_4^< = G^>(3, 3') G^>(4, 4') - G^<(3, 3') G^<(4, 4')$ is the two-particle spectral function which includes the Pauli blocking of internal states in the Galitskii–Feynman approximation. Both forms of $\text{Im } \mathcal{T}$ can be derived by algebraic manipulations with the ladder equation for $\mathcal{T}^{R,A}$ and hold out of equilibrium.

Implementation of the optical theorem (83) requires us to rearrange the scattering integrals. This is most conveniently done on the level of the starting nonequilibrium Green functions. The integrand in the right hand side of (9) is composed of two terms, in a reduced notation $I_l = G_1^> \Sigma_1^< - G_1^< \Sigma_1^>$ and $I_r = \Sigma_1^< G_1^> - \Sigma_1^> G_1^<$. For the Galitskii–Feynman selfenergy,

$$I_l = G_1^> G_2^> \mathcal{T}^R G_3^< G_4^< \mathcal{T}^A - G_1^< G_2^< \mathcal{T}^R G_3^> G_4^> \mathcal{T}^A, \quad (84)$$

we have to add and subtract $G_1^< G_2^< \mathcal{T}^R G_3^< G_4^< \mathcal{T}^A$ to the second term of (84) to achieve the Galitskii–Feynman two-particle spectral function,

$$I_l = \mathcal{A}_{12} \mathcal{T}^R G_3^< G_4^< \mathcal{T}^A - G_1^< G_2^< \mathcal{T}^R \mathcal{A}_{34} \mathcal{T}^A. \quad (85)$$

In the last term we can apply the optical theorem (83) for the interchange $\mathcal{T}^R \mathcal{A}_{34} \mathcal{T}^A \longrightarrow \mathcal{T}^A \mathcal{A}_{34} \mathcal{T}^R$; therefore

$$I_l = \mathcal{A}_{12} \mathcal{T}^R G_3^< G_4^< \mathcal{T}^A - G_1^< G_2^< \mathcal{T}^A \mathcal{A}_{34} \mathcal{T}^R. \quad (86)$$

In the same way one can rearrange I_r .

Technically, the interchange of the retarded and the advanced T-matrices is equivalent to the change of sign of the phase shift ϕ , $\mathcal{T}^R \rightarrow \mathcal{T}^R = |t| e^{-i\phi}$ and $\mathcal{T}^A \rightarrow \mathcal{T}^A = |t| e^{i\phi}$. The implementation of the optical theorem thus has no effect on the local parts of the scattering integral which depend exclusively on the amplitude $|t|$ of the T-matrix. In contrast, all nonlocal corrections given by Δ 's (73)–(79) are linear functions of the phase shift ϕ and thus reverse their signs. In the kinetic equation (80), the combination corresponding to \mathcal{A}_{34} is achieved by regrouping distributions as

$$(1 - f_1)(1 - \bar{f}_2) \bar{f}_3 \bar{f}_4 - f_1 \bar{f}_2 (1 - \bar{f}_3)(1 - \bar{f}_4) = (1 - f_1 - \bar{f}_2) \bar{f}_3 \bar{f}_4 - f_1 \bar{f}_2 (1 - \bar{f}_3 - \bar{f}_4). \quad (87)$$

The intermediate-state Pauli-blocking factor $1 - \bar{f}_3 - \bar{f}_4$ is the occupation number corresponding to \mathcal{A}_{34} . Inverting signs of the phase shift, i.e., of Δ 's, related to the scattering integral with the Pauli

blocking $1 - \bar{f}_3 - \bar{f}_4$ one arrives at the kinetic equation

$$\begin{aligned}
\frac{\partial f_1}{\partial t} + \frac{\partial \varepsilon_1}{\partial k} \frac{\partial f_1}{\partial r} - \frac{\partial \varepsilon_1}{\partial r} \frac{\partial f_1}{\partial k} &= \int \frac{dp}{(2\pi)^3} \frac{dq}{(2\pi)^3} \left(1 - \frac{1}{2} \frac{\partial \Delta_2}{\partial r} - \frac{\partial \varepsilon_2}{\partial r} \frac{\partial \Delta_2}{\partial \omega} \right) \\
&\times \left| t \left(\varepsilon_1 + \bar{\varepsilon}_2 - \Delta_E, k - \frac{1}{2} \Delta_K, p - \frac{1}{2} \Delta_K, q, r - \Delta_r, t - \frac{1}{2} \Delta_t \right) \right|^2 \\
&\times 2\pi \delta(\varepsilon_1 + \bar{\varepsilon}_2 - \bar{\varepsilon}_3 - \bar{\varepsilon}_4 - 2\Delta_E) (1 - f_1 - \bar{f}_2) \bar{f}_3 \bar{f}_4 \\
&- \int \frac{dp}{(2\pi)^3} \frac{dq}{(2\pi)^3} \left(1 + \frac{1}{2} \frac{\partial \Delta_2}{\partial r} + \frac{\partial \varepsilon_2}{\partial \tau} \frac{\partial \Delta_2}{\partial \omega} \right) \\
&\times \left| t \left(\varepsilon_1 + \varepsilon_2 + \Delta_E, k + \frac{1}{2} \Delta_K, p + \frac{1}{2} \Delta_K, q, r + \Delta_r, t + \frac{1}{2} \Delta_t \right) \right|^2 \\
&\times 2\pi \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + 2\Delta_E) f_1 f_2 (1 - f_3 - f_4). \tag{88}
\end{aligned}$$

In functions with bars arguments are given by (81); in functions without bars arguments are identical except for the reversed (i.e., positive) signs of all Δ 's.

The kinetic equation with the space-time symmetry of scattering integrals recalls the Enskog equation for classical hard spheres with respect to two features. First, if one suppresses the collision delay, see (82), the scattering-out becomes the space inversion of the scattering-in. Second, the scattering rate (given by $|t|^2$) is centered between initial and final states of collisions.

The kinetic equation (88) has a difficult interpretation because its scattering integrals have positive or negative values following the sign of the Pauli blocking factors $1 - f_1 - f_2 = (1 - f_1)(1 - f_2) - f_1 f_2$ and $1 - f_3 - f_4 = (1 - f_3)(1 - f_4) - f_3 f_4$. This strange behavior appears due to the stimulated transition processes given by the terms $-f_1 f_2$ and $-f_3 f_4$ on top of the standard Pauli blocking, $(1 - f_1)(1 - f_2)$ and $(1 - f_3)(1 - f_4)$. While in the local approximation of the scattering integrals the -in and -out stimulated transitions exactly compensate giving no net contribution; the nonlocal corrections translate this feature known from the equilibrium Green functions on the level of the kinetic equation. Our difficulties with the classical interpretation of (88) thus may follow from our low understanding of stimulated processes in general. At very low temperatures, the stimulated transitions become so strong that the above quasiparticle picture breaks down and the system develops super-conductivity having an additional degree of freedom and a different spectrum of single-particle excitations. Although the Galitskii–Feynman approximation is not suited to describe the super-conducting state, it reveals a singularity at T_c which signals its onset.

Let us put aside problems with stimulated processes and return to our discussion of characteristic times. For the moment we will assume that the collision delay Δ_t is positive. In the scattering-in integral which is the first term on the right hand side of (88), the initial condition is at the time $t - \Delta_t$, i.e., before the time t at which we balance the changes at the phase-space point (k, r) . This has a clear interpretation; at $t - \Delta_t$ the particle of momentum $k - q$ enters into the collision process from which it is released at t into (k, r) . In the scattering-out process, the particle leaves the cell (k, r) at t when it enters the collision process. This process will end at $t + \Delta_t$ when we have to check the accessible phase space. While the scattering-out has a natural interpretation, its description violates the causality. Indeed, the occupation factors in future time $t + \Delta_t$ are supposed to decide whether the process comes through or whether it is blocked. We note that the violation of the causality on the microscopic time scale is common in kinetic equations. Already the Fermi Golden Rule includes a time integral into future which eliminates off-shell processes. In fact, it is the Fermi-Golden-Rule-type approximation, in this paper represented by the pole approximation, which makes negative values of the collision delay possible.

For a detailed discussion of the space-time versus particle-hole symmetry for various forms of the two-particle spectral function see [70].

4.5. Correlated Density

As mentioned in the Introduction, in the system described on the level of the Galitskii–Feynman approximation the density of quasiparticles, $n_f = \int (dk/(2\pi)^3) f$, differs from the density of composing particles $n = \int (dk/(2\pi)^3) \rho$ by the amount called the correlated density $n_c = n - n_f$. After some algebra, from relation (22) one finds

$$n_c = \int \frac{dk dp dq}{(2\pi)^8} |t|^2 \Delta_t \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) f_1 f_2 (1 - f_3 - f_4). \quad (89)$$

The functions $|t|$, ε 's, and f 's are in the local approximation having no Δ 's in the arguments. This expression is the generalized Beth–Uhlenbeck formula [46, 48, 53, 59, 71–73]. The correlated density is proportional to the energy-derivative of the phase shift expressed here via the collision delay.

This correlated density is consistent with the equation of continuity found from the kinetic equation (88). For simplicity we will assume a homogeneous system for the equation of continuity reading $\delta_t n = 0$. Taking the momentum integral over (88) one finds

$$\frac{\partial}{\partial t} \int \frac{dk}{(2\pi)^3} f_1 = - \frac{\partial}{\partial t} \int \frac{dk dp dq}{(2\pi)^8} |t|^2 \Delta_t \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) f_1 f_2 (1 - f_3 - f_4). \quad (90)$$

Using (89) in the right hand side, one finds $\partial_t(n_f + n_c) = 0$. It should be noted that the number of quasiparticles is not conserved, $\partial_t n_f \neq 0$, because during the time interval Δ_t the colliding particles are excluded from the single-particle statistics being in two-particle scattering states.

One can see that instantaneous approximations of the kinetic equation cannot capture the correlated density. In the same time, the correct description of the quasiparticle density makes the choice (76) of the collision delay preferable to other characteristic times one can define in the spirit of Wigner from the asymptotic behavior of final states of binary collisions. In Appendix C we comment on some mistakes done in previous studies of the correlated density.

The complete proof of conservation laws with explicit expressions for the correlated pressure and energies can be found in [52].

5. SUMMARY

In absence of the time operator, there are many definitions of characteristic times one can associate with collisions. Regardless of a definition one uses, if the finite duration of collisions is included in a kinetic equation, this equation is non-Markovian since the initial and final states of the collision are at distinct times. The family of non-Markovian kinetic equations also includes retarded equations of Levinson type in which the collision is expressed in terms of the time integral describing the whole process of the two-particle interaction.

For slowly varying systems, we have shown that one-half of the Levinson-type retardation describes the off-shell motion and corresponding renormalizations of single-particle functions. The second half compensates for the decay of propagators during the integration and can be eliminated leaving the pole contribution to the scattering integral identical to the one obtained from the quasiparticle approximation.

Finally, we have discussed the collision delay resulting from the energy dependence of the scattering phase shift. We have shown that this non-Markovian correction is connected with other nonlocal corrections to the scattering integral which gives us a freedom to define the collision duration in different ways. The physical quantity sensitive to the actual choice of the collision delay is the density

of quasiparticles, or its complementary quantity, the density of correlated particles. The method presented in this paper, and at the same time the choice of the collision delay, reproduces the correlated density obtained within the generalized Beth–Uhlenbeck approach.

The noninstant and nonlocal corrections given by the Δ 's do not change the structure and the overall interpretation of the scattering integral but only slightly renormalize its ingredients. The exclusive dependence of the nonlocal and noninstant corrections on the scattering phase shift confirms results from the theory of gases [28, 29, 32, 37] obtained by very different technical tools. The noninstant and nonlocal scattering integral in the form (88) parallels the classical Enskog equation; therefore it can be treated with numerical tools developed for the theory of classical gases; see, e.g., [74] at least as long as $1 - f_1 - f_2$ remains positive.

The virial corrections to the balance equations appear from intrinsic gradients instead of correlation parts in the equation of the reduced density matrix. In [52] it is shown that the nonlocal kinetic equation leads to complete balance equations for the density, energy, and stress tensor which establish conservation laws including correlated parts. The nonlocal kinetic equation derived here requires no more numerical effort than solving the Boltzmann equation. The numerical solution and application has been demonstrated in [75–77].

APPENDIX A. T-MATRIX APPROXIMATION

In order to describe short-ranged two-particle interactions, it is necessary to introduce the standard approximation of the many-body theory, the T-matrix approximation [54, 66, 78]. In the following we give a brief compilation of important formulas.

The sum of ladder diagrams is defined as the causal T-matrix

$$\langle 12|\mathcal{T}|1'2'\rangle = \mathcal{V}(12|1'2') + \mathcal{V}(1233')G(3\bar{1})G(3'\bar{2})\langle \bar{1}\bar{2}|\mathcal{T}|1'2'\rangle. \quad (\text{A.1})$$

Arguments not present on the left hand side are integrated over. For simplicity, we do not assume the sum over spin and isospin and the anti-symmetrization for identical particles. Since the potential is local in time, we can simplify the T-matrix as $\langle 12|\mathcal{T}|1'2'\rangle = \langle x_1x_2t_1|\mathcal{T}|x'_1x'_2t'_1\rangle\delta(t_1 - t_2)\delta(t'_1 - t'_2)$, and ladder summation (A.1) reads

$$\langle x_1x_2t|\mathcal{T}|x'_1x'_2t'\rangle = \mathcal{V}(x_1x_2x'_1x'_2)\delta(t - t') + \mathcal{V}(x_1x_2x_3x'_3)\langle x_3x'_3t|\mathcal{G}|\bar{x}_1\bar{x}_2\bar{t}\rangle\langle \bar{x}_1\bar{x}_2\bar{t}|\mathcal{T}|x'_1x'_2t'\rangle, \quad (\text{A.2})$$

where we have introduced the two-particle Green function,

$$\langle x_1x_2t|\mathcal{G}|\bar{x}_1\bar{x}_2\bar{t}\rangle = G(x_1t\bar{x}_1\bar{t})G(x_2t\bar{x}_2\bar{t}). \quad (\text{A.3})$$

The causal selfenergy then reads

$$\Sigma(11') = -i \int dx_2 dx'_2 G(x'_2t'_1x_2t_1^+) \langle x_1x_2t_1|\mathcal{T}|x'_1x'_2t'_1\rangle. \quad (\text{A.4})$$

Using the Langreth–Wilkins rules [79] (which are equivalent to the algebra on the Keldysh contour) we obtain the real-time Green's functions. From (A.4) follows

$$\begin{aligned} \Sigma^{\cong} &= \mathcal{T}^{\cong} G^{\cong} \\ \Sigma^R &= -i\Theta(t_1 - t_2)[\mathcal{T}^> G^< + \mathcal{T}^< G^>] \\ &= \mathcal{T}^R G^< - \mathcal{T}^< G^A. \end{aligned} \quad (\text{A.5})$$

We have abbreviated the notation; all integrations and variables are the same as in (A.4). From (A.2) follows [55, 66, 80]

$$\mathcal{T}^{\gtrless} = \mathcal{T}^R \mathcal{G}^{\gtrless} \mathcal{T}^A \quad (\text{A.6})$$

$$\mathcal{T}^{R/A} = \mathcal{V} + \mathcal{V} \mathcal{G}^{R/A} \mathcal{T}^{R/A}. \quad (\text{A.7})$$

Explicitly (A.6) reads

$$\begin{aligned} & \langle x_1 x_2 t | \mathcal{T}_{ab}^{\gtrless} | x'_1 x'_2 t' \rangle \\ &= \int d\bar{x}_1 d\bar{x}'_1 d\bar{x}_2 d\bar{x}'_2 d\bar{t} d\bar{t}' \langle x_1 x_2 t | \mathcal{T}_{ab}^R | \bar{x}_1 \bar{x}_2 \bar{t} \rangle \langle \bar{x}_1 \bar{x}_2 \bar{t} | \mathcal{G}_{ab}^{\gtrless} | \bar{x}'_1 \bar{x}'_2 \bar{t}' \rangle \langle \bar{x}'_1 \bar{x}'_2 \bar{t}' | \mathcal{T}_{ab}^A | x'_1 x'_2 t' \rangle. \end{aligned} \quad (\text{A.8})$$

In the scattering integral we use the following representation of the two-particle T-matrix,

$$\begin{aligned} & \langle x_1 x_2 \bar{t} | \mathcal{T} | x'_1 x'_2 t' \rangle \\ &= \int \frac{dk dp dq d\Omega}{(2\pi)^{10}} \mathcal{T}(\Omega, k, p, q, r, t) \exp[i(kx_1 + px_2 - (k - q)x'_1 - (p + q)x'_2 - \Omega(\bar{t} - t'))], \end{aligned} \quad (\text{A.9})$$

where $r = \frac{1}{4}(x_1 + x_2 + x'_1 + x'_2)$ and $t = \frac{1}{2}(\bar{t} + t')$ are center-of-the-mass coordinate and time.

A.1. Quasiparticle Approximation

We introduce the notation for the known quasiparticle approximation without gradient corrections. In the quasiparticle approximation the free two-particle propagator \mathcal{G} (denoted in the mixed Wigner representation as ζ) takes the form

$$\zeta^R(\Omega, p, p', r, t) = \frac{1 - f(p, r, t) - f(p', r, t)}{\Omega - \varepsilon(p, r, t) - \varepsilon(p', r, t) + i\eta}. \quad (\text{A.10})$$

The selfenergy then reads

$$\sigma^>(\omega, k, r, t) = \int \frac{dp}{(2\pi)^3} t^>(\omega + \varepsilon(p, r, t), k, p, 0) f(p, r, t), \quad (\text{A.11})$$

and analogously $\sigma^<$. In the rest of this section we suppress variables r, t . From (A.7) one finds

$$t^<(\Omega, k, p, 0) = \int \frac{dq}{(2\pi)^3} |t^R|^2(\varepsilon_{k-q} + \varepsilon_{p+q}, k, p, q) f_{k-q} f_{p+q} 2\pi \delta(\omega - \varepsilon_{p+q} - \varepsilon_{k-q}), \quad (\text{A.12})$$

and $t^>$ is given by interchange $f \leftrightarrow 1 - f$.

For the determination of the quasiparticle energy one needs the retarded selfenergy

$$\begin{aligned} \sigma^R(\omega, k) &= - \int \frac{dp}{(2\pi)^3} \frac{d\omega'}{2\pi} \frac{t^>(\omega' + \varepsilon_p, k, p, 0) f_p}{\omega' - \omega - i\eta} - \int \frac{dp}{(2\pi)^3} \frac{d\omega'}{2\pi} \frac{t^<(\omega' + \varepsilon_p, k, p, 0)(1 - f_p)}{\omega' - \omega - i\eta} \\ &= \int \frac{dp}{(2\pi)^3} t^R(\omega + \varepsilon_p, k, p, 0) f_p - \int \frac{dp dq}{(2\pi)^6} \frac{|t^R(\varepsilon_{k-q} + \varepsilon_{p+q}, k, p, q)|^2 f_{k-q} f_{p+q}}{\varepsilon_{k-q} + \varepsilon_{p+q} - \varepsilon_p - \omega + i\eta}. \end{aligned} \quad (\text{A.13})$$

In the real part of the selfenergy the real part of the T-matrix appears,

$$\text{Re}t^R(\Omega, k, p, 0) = \int \frac{dq}{(2\pi)^3} \frac{1 - f_{k-q} - f_{p+q}}{\Omega - \epsilon_{k-q} - \epsilon_{p+q}} |t^R|^2 (\epsilon_{k-q} + \epsilon_{p+q}, k, p, q). \quad (\text{A.14})$$

This form follows from the Kramers–Kronig transformation of $t^> - t^<$.

The ladder equation without gradients reads (notation is specified by (B.2))

$$\langle p_1 | t^R(q') | p_2 \rangle = \langle p_1 | \mathcal{V}(q') | p_2 \rangle + \int \frac{dp' dp''}{(2\pi)^6} \langle p_1 | \mathcal{V}(q') | p' \rangle \langle p' | \zeta^R(q') | p'' \rangle \langle p'' | t_{sc}^R(q') | p_2 \rangle. \quad (\text{A.15})$$

In the coordinates, $k = \frac{q'}{2} + p_1$, $p = \frac{q'}{2} - p_1$, $q = p_1 - p_2$, corresponding to (A.9) this equation takes the form

$$\begin{aligned} t^R(\Omega, k, p, q) &= \mathcal{V}(k, p, q) + \int \frac{dp' dp''}{(2\pi)^6} \mathcal{V}(k, p, p') \zeta^R(\Omega, k - p', p + p', p'' - p') \\ &\quad \times t^R(\Omega, k - p'', p + p'', q - p''). \end{aligned} \quad (\text{A.16})$$

Since $\zeta^{\cong}(k, p, q) = (2\pi)^3 \delta(q) g^{\cong}(k) g^{\cong}(p)$, in the quasiparticle approximation one obtains

$$\begin{aligned} t^R(\Omega, k, p, q) &= \mathcal{V}(k, p, q) + \int \frac{dq'}{(2\pi)^3} \mathcal{V}(k, p, q') \frac{1 - f_{k-q'} - f_{p+q'}}{\Omega - \epsilon_{p+q'} - \epsilon_{k-q'} + i\eta} t^R(\Omega, k - q', p + q', q - q'). \end{aligned} \quad (\text{A.17})$$

APPENDIX B. GRADIENT EXPANSION

Let us consider a product of two-particle functions

$$\mathcal{C}(1234) = \int d3' d4' \mathcal{A}(123'4') \mathcal{B}(3'4'34). \quad (\text{B.1})$$

We transform this product into mixed representation (A.9) keeping gradients until the linear order.

Variables in (A.9) follow variables of the scattering integral. For the scattering-in, the momenta correspond to the process $k - q, p + q \rightarrow k, p$. These variables, however, do not form a convenient algebra for the gradient expansion. To this end we use the coordinates

$$\begin{aligned} \alpha &= 1 - 2 \\ \beta &= 3 - 4 \\ \tau &= \frac{1}{2}(1 + 2 - 3 - 4) \\ x &= \frac{1}{4}(1 + 2 + 3 + 4) \equiv (r, t). \end{aligned} \quad (\text{B.2})$$

Since the relative coordinates, α and β , obey the standard matrix algebra while only the coordinates τ and x undergo the gradient expansion, we write these arguments in the form $\langle\alpha|C(\tau, x)|\beta\rangle$. In this notation the product (B.1) reads

$$\langle\alpha|C(\tau, x)|\beta\rangle = \int d\gamma d\bar{\tau} \langle\alpha|A\left(\tau - \bar{\tau}, x + \frac{1}{2}\bar{\tau}\right)|\gamma\rangle \langle\gamma|B\left(\bar{\tau}, x + \frac{1}{2}(\tau - \bar{\tau})\right)|\beta\rangle. \quad (\text{B.3})$$

Assuming a slow variation of the center-of-mass coordinate x we can write this product keeping gradients in x up to the linear order. The relative coordinates can be suppressed being independent of this expansion. Via the Fourier transformation of coordinate τ ,

$$C(\tau, x) = \int \frac{d\kappa}{(2\pi)^4} c(\kappa, x) e^{i(\sum_{n=1}^3 \kappa_n \tau_n - i\kappa_4 \tau_4)}, \quad (\text{B.4})$$

we express the operator C as a function of the sum momentum, $(\kappa_1, \kappa_2, \kappa_3) = k + p$, and energy $\kappa_4 = \Omega$ of both particles. One can see that the product has the same form as products of single-particle functions,

$$c = ab - \frac{i}{2}[a, b] = ab\left(1 - \frac{i}{2}[\ln a, \ln b]\right), \quad (\text{B.5})$$

where the (generalized) Poisson bracket applies to the sum coordinate and time. The logarithmic form of (B.5) is merely a convenient notation because a and b are matrices in α and β . The logarithm is taken of each matrix element of a and b and summed together with the product ab .

In parallel with the Dyson equation, there are no linear gradients in the ladder equation (A.7). Like the retarded Green function, the T-matrix is given by the matrix inversion, $T_R^{-1} = \mathcal{V}^{-1} - \mathcal{G}^R$. Since the T-matrix is symmetric with respect to matrix arguments, $\langle s|\mathcal{T}|s'\rangle = \langle s'|\mathcal{T}|s\rangle$ (s and s' are momenta associated with α and β , respectively), the matrix inversion does not bring any gradients.

The selfenergy, $\Sigma^<(13) = T^<(1234)G^>(42)$, includes a number of gradient contributions due to internal gradients in $T^<$ and a simple nonlocal correction due to the convolution with $G^>$.

B.1. Two-Particle Matrix Products

For the calculation of the selfenergy in (B.15) we need $t^<$ for the zero transferred momentum, $q = 0$, and its infinitesimal vicinity. Making the gradient expansion of (A.6) in terms of the Poisson brackets one finds

$$\begin{aligned} t^< &= t^R \tilde{g}^< t^A \left(1 - \frac{i}{2}([\ln t^R, \ln \tilde{g}^<] - [\ln t^A, \ln \tilde{g}^<] + [\ln t^R, \ln t^A])\right) \\ &= |t|^2 \tilde{g}^< (1 - [\phi, \ln \tilde{g}^<] - [\phi, \ln |t|]). \end{aligned} \quad (\text{B.6})$$

In the second line we have decomposed the T-matrices into amplitudes and phase shifts, $t^{R,A} = |t|e^{\mp i\phi}$. Note that all gradient corrections depend exclusively on the derivatives of phase shift ϕ . Using definitions (73)–(79) and the linear approximation, $a(x)(1 - \Delta\partial_x a) = a(x - \Delta)$, expression

(B.6) can be given a form

$$\begin{aligned}
& t^<\left(\Omega, k, p, 0, r - \frac{1}{2}\Delta_2, t\right) \\
&= \int \frac{dq}{(2\pi)^3} \frac{dQ}{(2\pi)^3} t^R\left(\Omega - \Delta_\omega, k - \frac{\Delta_k}{2}, p - \frac{\Delta_k}{2}, q, r - \Delta_r, t - \frac{\Delta t}{2}\right) \\
&\quad \times \zeta^<\left(\Omega - 2\Delta_\omega, k - q - \Delta_k, p + q - \Delta_k, Q, r - \frac{\Delta_3}{2} - \frac{\Delta_4}{2}, t - \Delta_t\right) \\
&\quad \times t^A\left(\Omega - \Delta_\omega, k - \frac{\Delta_k}{2}, p - \frac{\Delta_k}{2}, q + Q, r - \Delta_r, t - \frac{\Delta_t}{2}\right). \tag{B.7}
\end{aligned}$$

Factors of half in the nonlocal corrections to the amplitude $|t^{R,A}|$ result from the fact that this amplitude enters the collision integral in the square.

In the absence of gradients, the two-particle function $\zeta^<$ is singular in the “transferred” momentum Q giving the only contribution for $Q=0$. When the gradient corrections are already in the explicit form, one can employ this symmetry as it has been used above. Indeed, the complex conjugacy of the advanced and retarded T-matrices, $t^A = \bar{t}^R$, requires equal arguments of both functions. Now we show that the infinitesimal vicinity of $Q=0$ brings additional gradient contributions.

B.2. Convolution of Initial States

The two-particle correlation function $\zeta^<$ representing initial states of the collision in (B.7) is not a suitable input for the kinetic equation. We have to express $\zeta^<$ in terms of the convolution of two single-particle functions. This convolution brings gradient corrections by which effective positions of particles entering the collision process become distinct.

From the inverse transformation to (A.9) we obtain

$$\begin{aligned}
& \zeta^<(\Omega, k - p, p + q, Q, r, t) \\
&= 2^3 \int dx_1 dx_2 dx'_1 dx'_2 d\bar{t} dt' \exp[iQ(x'_1 - x'_2) + i\Omega(\bar{t} - t')] \\
&\quad \times \exp[-i(k - q)(x_1 - x'_1) - i(p + q)(x_2 - x'_2)] \\
&\quad \times \delta(x_1 + x_2 + x'_1 + x'_2 - 4r)\delta(\bar{t} + t' - 2t) G^<(x_1\bar{t}, x'_1 t') G^<(x_2\bar{t}, x'_2 t'). \tag{B.8}
\end{aligned}$$

The substitution, $x_1 = r + \alpha + \beta_1/2$, $x'_1 = r + \alpha - \beta_1/2$, $x_2 = r - \tilde{\alpha} + \beta_2/2$, $x'_2 = r - \tilde{\alpha} - \beta_2/2$, $\bar{t} = t + \tau/2$, $t' = t - \tilde{\tau}/2$, shows that the δ -functions yield $\tilde{\tau} = \tau$ and $\tilde{\alpha} = \alpha$ and we obtain

$$\begin{aligned}
& \zeta^<(\Omega, k - p, p + q, Q, r, t) \\
&= 2^3 \int d\beta_1 d\beta_2 d\alpha d\tau \exp[2iQ\alpha] \exp[-i(k - q + Q/2)\beta_1 \\
&\quad - i(p + q - Q/2)\beta_2 + i\Omega\tau] \hat{g}^<(\beta_1, r + \alpha, t, \tau) \hat{g}^<(\beta_2, r - \alpha, t, \tau), \tag{B.9}
\end{aligned}$$

where the representation (9) has been used.

Now we linearize the center-of-mass dependence of $g^<$'s in α . The factor α can be represented by a derivative in front of the integral, $\alpha = \frac{i}{2}\partial_Q - \frac{i}{4}\partial_q$. The remaining integration over α results in $\pi^3\delta(Q)$. When substituted into (B.7), the differential operator in front of the integral is treated with

the help of the integration by parts giving

$$\alpha \rightarrow -\frac{1}{2}(\Delta_3 - \Delta_4) \quad (\text{B.10})$$

and Q is integrated out. In other words the α in the arguments of \hat{g} in (B.9) can be replaced by nonlocal shifts valid up to linear orders in gradients. This leads with (B.7) to

$$\begin{aligned} & t^< \left(\Omega, k, p, 0, r - \frac{1}{2}\Delta_2, t \right) \\ &= \int \frac{dq}{(2\pi)^3} \left| t^R \left(\Omega - \Delta_\omega, k - \frac{\Delta_k}{2}, p - \frac{\Delta_k}{2}, q, r - \Delta_r, t - \frac{\Delta_t}{2} \right) \right|^2 \\ & \quad \times g^<(k - q, r - \Delta_3, t - \Delta_t, \Omega - 2\Delta_\omega) g^<(p + q, r - \Delta_4, t - \Delta_t, \Omega - 2\Delta_\omega). \end{aligned} \quad (\text{B.11})$$

B.3. Convolution of \mathcal{T} -Matrix and Hole Green Function

Next we evaluate the convolution of the \mathcal{T} -matrix with the hole Green function which is required for the selfenergy. Writing the selfenergy in the above representation [$\tau = 1 - 3$ and $x = \frac{1}{2}(1 + 3)$],

$$\begin{aligned} \Sigma^<(\tau, x) &= \int d\alpha d\beta G^> \left(-\tau - \beta + \alpha, x - \frac{1}{2}(\alpha + \beta) \right) \\ & \quad \times \langle \alpha | \mathcal{T}^< \left(\tau - \frac{1}{2}(\alpha - \beta), x - \frac{1}{4}(\alpha + \beta) \right) | \beta \rangle, \end{aligned} \quad (\text{B.12})$$

we see that the convolution couples matrix arguments α and β with the center-of-mass variables. By substitution, $\lambda = \frac{1}{2}(\alpha + \beta)$ and $\mu = \alpha - \beta$, and expansion in gradients we obtain

$$\begin{aligned} \Sigma^<(\tau, x) &= \int d\mu G^>(\mu - \tau, x) \int d\lambda \left\langle \lambda + \frac{\mu}{2} \left| \mathcal{T}^< \left(\tau - \frac{\mu}{2}, x \right) \right| \lambda - \frac{\mu}{2} \right\rangle \\ & \quad - \frac{1}{2} \int d\mu G^>(\mu - \tau, x) \frac{\partial}{\partial x} \int d\lambda \lambda \left\langle \lambda + \frac{\mu}{2} \left| \mathcal{T}^< \left(\tau - \frac{\mu}{2}, x \right) \right| -\lambda - \frac{\mu}{2} \right\rangle \\ & \quad - \int d\mu \left(\frac{\partial}{\partial x} G^>(\mu - \tau, x) \right) \int d\lambda \lambda \left\langle \lambda + \frac{\mu}{2} \left| \mathcal{T}^< \left(\tau - \frac{\mu}{2}, x \right) \right| \lambda - \frac{\mu}{2} \right\rangle. \end{aligned} \quad (\text{B.13})$$

The second and third terms are gradient corrections due to the convolution.

Now we can transform the selfenergy (B.13) into the mixed representation. The momentum representation of the matrix algebra is introduced via unity operators, e.g., $1 = \int (ds/(2\pi)^3) |s\rangle \langle s|$, with $\langle \lambda - \mu/2 | s \rangle = e^{is(\lambda - \mu/2)}$ and $\langle s' | \lambda + \mu/2 \rangle = e^{-is'(\lambda + \mu/2)}$. For the nongradient part the integration over λ results in the known fact that only the diagonal element, $s = s'$, contributes. For the gradient contribution, $\lambda \rightarrow -(i/2)(\partial_s - \partial_{s'})$, and the diagonal element is taken after the derivatives are performed. Since we have considered the gradient corrections to the \mathcal{T} -matrix already in the last section and since the shifts are additive, it is sufficient here to use the zero-order approximation of $t^<$,

$$\langle s | t^< | s' \rangle = \int \frac{d\bar{s} d\bar{s}'}{(2\pi)^2} \langle s | t^R | \bar{s} \rangle \langle \bar{s} | \zeta^< | \bar{s}' \rangle \langle \bar{s}' | t^A | s' \rangle, \quad (\text{B.14})$$

where all functions have the center-of-mass argument (κ, x) . From $t^{R,A} = |t| e^{\mp i\phi}$ and condition $s = s'$ we find that $\lambda \rightarrow -\partial_s \phi$. By the substitution into variables of the kinetic equation, $(q, 0) = s - s'$, $(k, \omega) = \kappa/2 + s$, $(p, 0) = \kappa/2 - s$ and $(r, t) = x$, one confirms that $-\partial_s \phi = \frac{1}{2}\Delta_2$, see (73),

and the selfenergy reads

$$\sigma_{\omega}^{<}(k, r, t) = \int \frac{dp d\Omega}{(2\pi)^4} g_{\Omega-\omega}^{>}(p, r - \Delta_2, t) \left(1 - \frac{1}{2} \frac{\partial \Delta_2}{\partial r}\right) t^{<} \left(\Omega, k, p, 0, r - \frac{1}{2} \Delta_2, t\right). \quad (\text{B.15})$$

In this expression, $t^{<}$ abbreviates the right hand side of (B.14), because the nonlocal correction Δ_2 is defined only with respect to the integral over internal states of the collision. The norm term, $1 - \frac{1}{2} \partial_r \Delta_2$, results from the interchange, $\partial_r(\Delta_2 A) = (\partial_r \Delta_2)A + \Delta_2 \partial_r A$, one has to make before the derivative is expressed in terms of the displacement, e.g., $g^{<}(r) - \Delta_2 \partial_r g^{<}(r) = g^{<}(r - \Delta_2)$.

Together with (B.11) we obtain the result (72).

APPENDIX C. COMMON PITFALLS WITH THE CORRELATED DENSITY

We note that one of us made a mistake deriving the correlated density directly from the retardation in the Levinson-type equation [59]. The expression found in [59] is identical to the correlated density except for two points: there is a wrong sign and the amplitude is twice as large. Since this is a pitfall often met in the literature we want to give some details here so that it can be avoided in the future.

The simple expansion of the Levinson equation (3) up to the first order in memory [59] yields

$$\frac{\partial}{\partial t} \rho = \mathcal{I} + \frac{\partial}{\partial t} \mathcal{R}. \quad (\text{C.1})$$

To obtain balance equations for the density including virial corrections, one has to integrate (C.1) over momentum k ; see [59, 81]. Following the arguments developed for the Boltzmann-type equations, in [59], the wrong assumption was used that the correlated part of the collision integral, $\tilde{n}_c = \int dk R$, combines with the left hand side of (C.1), $\tilde{n}_f = \int dk \rho$, to establish the density conservation, $\frac{\partial}{\partial t}(\tilde{n}_f + \tilde{n}_c) = 0$. The correlated density derived in this way reminds us of the result known from equilibrium [46, 48, 71–73]. Note that in this picture, the Wigner distribution in the left hand side of the Levinson equation is treated as the quasiparticle distribution, a confusion occurring usually in density operator studies [81, 82]. The wrong sign follows from this last misinterpretation. By definition the momentum integral over the Wigner distribution yields the full density, $\int dk \rho = n$. The conservation law thus tells us that either $\int dk R = 0$ or the Levinson equation does not conserve the number of particles.

As we have shown in this paper the Levinson equation contains additional gradient terms which exactly compensate the explicit time gradients (used to derive \tilde{n}_f). The above double count follows from the double count of the off-shell contributions discussed in this paper.

Finally, the $\tau_{R,A}$ retardation was neglected. Since the correlated density (89) found from the off-shell contribution to the Wigner distribution is consistent with the conservation of the number of particles found from the $\tau_{R,A}$ retardation, one can see that these two contributions cancel and the collision integral of the Levinson equation conserves the number of particles, as it should.

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